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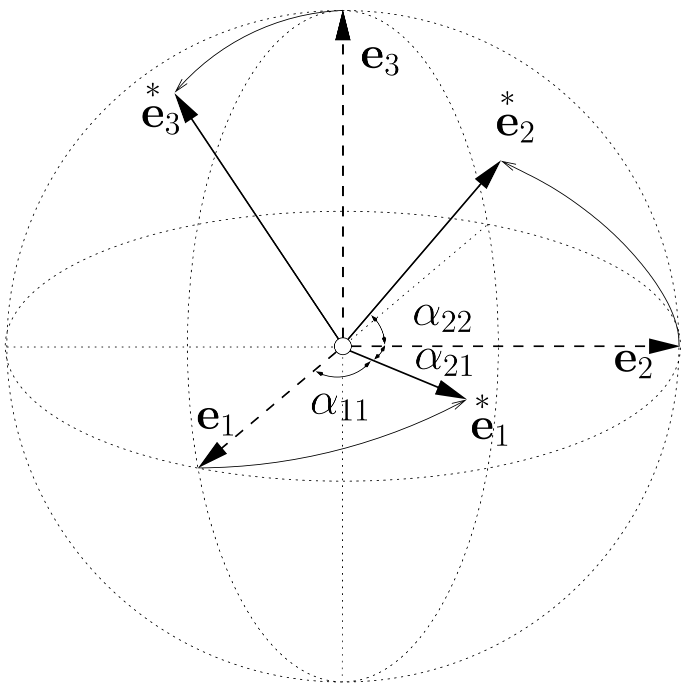
Vector and Tensor Calculus
An Introduction



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Contents

1 Mathematical Prerequisites

1.1 Basics of vector calculus

(a) Symbols, summation convention, Kronecker δ

Single- or multiple subscripts

$$
\begin{array}{lll}
u_i & \longrightarrow & u_1,\ u_2,\ u_3,\ \dots \\
u_i\, v_k & \longrightarrow & u_1\, v_1,\ u_1\, v_2,\ u_1\, v_3,\ \dots \\
 & & u_2\, v_1,\ u_2\, v_2,\ \dots \\
 & & \dots \\
t_{ik} & \longrightarrow & t_{11},\ t_{12},\ \dots \\
 & & \dots
\end{array}
$$

Summation convention of Einstein

Definition: Whenever the same subscript occurs twice in a term, a summation over that "double" subscript has to be carried out.

Example:

$$
\begin{aligned}
u_j\, v_j &= u_1\, v_1, + u_2\, v_2, + \dots + u_n\, v_n, \\
&= \sum_{j=1}^{n} u_j\, v_j
\end{aligned}
$$

Kronecker **symbol**

Definition: It exists a symbol δ_{ik} with the following properties

$$
\delta_{ik} = \begin{cases} 0 \text{ if } i \neq k \\ 1 \text{ if } i = k \end{cases}
$$

Example:

$$
u_i\, \delta_{ik} = u_1\, \delta_{1k} + u_2\, \delta_{2k} + \dots + u_n\, \delta_{nk}
$$

$$
\text{with} \quad u_1\, \delta_{1k} = \begin{cases} u_1\, \delta_{11} = u_1 \\ u_1\, \delta_{12} = 0 \\ \vdots \\ u_1\, \delta_{1n} = 0 \end{cases}
$$

$$
\longrightarrow \quad u_i\, \delta_{ik} = u_k
$$

If the Kronecker symbol is multiplied with another quantity and if there is a double subscript in this term, the Kronecker symbol disappears, the "double" subscript can be dropped and the free subscript remains.

Rem.: Subscripts occuring two times in a term can be renamed arbitrarily.

(b) Terms and definitions of vector algebra

Rem.: The following statements are related to the standard three-dimensional (3-d) physical space, i. e. the EUCLIDean vector space $\mathcal{V}^3$.
Generally, SPACE is a mathematical concept of a set and does not directly refer to the 3-d point space $\mathcal{E}^3$ and the 3-d vector space $\mathcal{V}^3$.

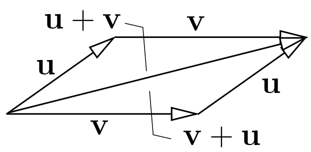
A: Vector addition

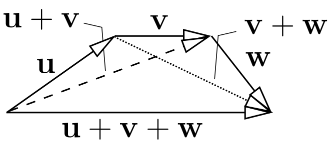
Requirement: $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, ...\} \in \mathcal{V}^3$

The following relations hold:

$$\begin{aligned} \mathbf{u}+\mathbf{v} &= \mathbf{v}+\mathbf{u} && : \text{commutative law} \\ \mathbf{u}+(\mathbf{v}+\mathbf{w}) &= (\mathbf{u}+\mathbf{v})+\mathbf{w} && : \text{associative law} \\ \mathbf{u}+\mathbf{0} &= \mathbf{u} && : \ \mathbf{0} \ : \text{identity element of vector addition} \\ \mathbf{u}+(-\mathbf{u}) &= \mathbf{0} && : \ -\mathbf{u} \ : \text{inverse element of vector addition} \end{aligned}$$

Examples to the commutative and the associative law:





B: Multiplication of a vector with a scalar quantity

Requirement: $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, ...\} \in \mathcal{V}^3; \ \{\alpha, \beta, ...\} \in \mathbb{R}$

$$\begin{aligned} 1\,\mathbf{v} &= \mathbf{v} && : 1\text{: identity element} \\ \alpha\,(\beta\,\mathbf{v}) &= (\alpha\,\beta)\,\mathbf{v} && : \text{associative law} \\ (\alpha+\beta)\,\mathbf{v} &= \alpha\,\mathbf{v}+\beta\,\mathbf{v} && : \text{distributive law (addition of scalars)} \\ \alpha\,(\mathbf{v}+\mathbf{w}) &= \alpha\,\mathbf{v}+\alpha\,\mathbf{w} && : \text{distributive law (addition of vectors)} \\ \alpha\,\mathbf{v} &= \mathbf{v}\,\alpha && : \text{commutative law} \end{aligned}$$

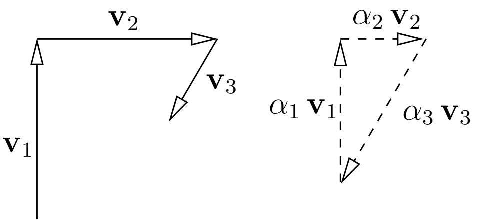
Rem.: In the general vector calculus, the definitions A and B constitute the “affine vector space”.

Linear dependency of vectors

Rem.: In $\mathcal{V}^3$, 3 non-coplanar vectors are linearly independent; i. e. each further vector can be expressed as an multiple of these vectors.

> **Theorem:** The vectors $\mathbf{v}_i$ $(i = 1, 2, 3, ..., n)$ are linearly dependent, if real numbers α_i exist which are not all equal to zero, such that
>
> $$\alpha_i \, \mathbf{v}_i = \mathbf{0} \quad \text{or} \quad \alpha_1 \, \mathbf{v}_1 + \alpha_2 \, \mathbf{v}_2 + ... + \alpha_n \, \mathbf{v}_n = \mathbf{0}$$

Example (plane case):



$\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 \neq \mathbf{0}$

but: $\alpha_1 \, \mathbf{v}_1 + \alpha_2 \, \mathbf{v}_2 + \alpha_3 \, \mathbf{v}_3 = \mathbf{0}$

$\longrightarrow$ $\{\mathbf{v}_1, \, \mathbf{v}_2, \, \mathbf{v}_3\}$: linearly dependent

$\longrightarrow$ $\{\mathbf{v}_1, \, \mathbf{v}_2\}$: linearly independent

Rem.: The α_i can be multiplied by any factor λ.

Basis vectors in $\mathcal{V}^3$

ex. : $\{\mathbf{v}_1, \, \mathbf{v}_2, \, \mathbf{v}_3\}$: linearly independent

then : $\{\mathbf{v}_1, \, \mathbf{v}_2, \, \mathbf{v}_3, \, \mathbf{v}\}$: linearly dependent

Thus, it follows that

$$\alpha_1 \, \mathbf{v}_1 + \alpha_2 \, \mathbf{v}_2 + \alpha_3 \, \mathbf{v}_3 + \lambda \, \mathbf{v} = \mathbf{0}$$

$$\longrightarrow \quad \lambda \, \mathbf{v} = -\alpha_i \, \mathbf{v}_i$$

$$\text{or} \quad \mathbf{v} = \frac{-\alpha_i}{\lambda} \, \mathbf{v}_i =: \beta_i \, \mathbf{v}_i$$

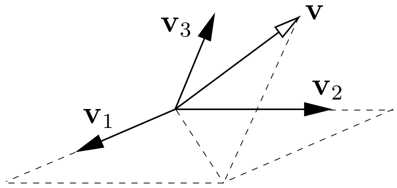
$$\text{with} \left\{ \begin{array}{lll} \beta_i = \dfrac{-\alpha_i}{\lambda} & : & \text{coefficients (of the vector components)} \\ \mathbf{v}_i & : & \text{basis vectors of } \mathbf{v} \end{array} \right.$$

Choice of a specific basis

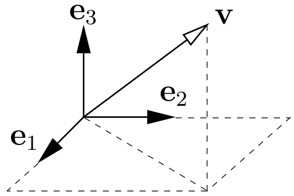
Rem.: In $\mathcal{V}^3$, each system of 3 linearly independent vectors can be selected as a basis; e. g.

$\mathbf{v}_i$: general basis

$\mathbf{e}_i$: specific, orthonormal basis (Cartesian, right-handed)



Basissystem $\mathbf{v}_i$



Basissystem $\mathbf{e}_i$

Representation of the vector $\mathbf{v}$:

$$\mathbf{v} = \begin{cases} \beta_i \, \mathbf{v}_i \\ \gamma_i \, \mathbf{e}_i \end{cases}$$

here: Specific choice of the Cartesian basis system $\mathbf{e}_i$

Notations

$$\mathbf{v} \;=\; v_i \, \mathbf{e}_i \;=\; v_1 \, \mathbf{e}_1 + v_2 \, \mathbf{e}_2 + v_3 \, \mathbf{e}_3$$

$$\text{with} \quad \begin{cases} v_i \, \mathbf{e}_i & : \ \text{vector components} \\ v_i & : \ \text{coefficients of the vector components} \end{cases}$$

C: Scalar product of vectors

The following relations hold:

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= \mathbf{v} \cdot \mathbf{u} && : \text{commutative law} \\ \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} && : \text{distributive law} \\ \alpha \, (\mathbf{u} \cdot \mathbf{v}) &= \mathbf{u} \cdot (\alpha \, \mathbf{v}) = (\alpha \, \mathbf{u}) \cdot \mathbf{v} && : \text{associative law} \\ \mathbf{u} \cdot \mathbf{v} &= 0 \qquad \forall \, \mathbf{u}, \text{ if } \ \mathbf{v} \equiv \mathbf{0} \\ \longrightarrow \mathbf{u} \cdot \mathbf{u} &\neq 0 \qquad , \text{ if } \ \mathbf{u} \neq \mathbf{0} \end{aligned}$$

Rem.: The definitions A, B and C constitute the "EUCLIDean vector space". If instead of $\mathbf{u} \cdot \mathbf{u} \neq 0$ especially

$$\mathbf{u} \cdot \mathbf{u} > 0 \ \ , \text{ if } \ \mathbf{u} \neq \mathbf{0},$$

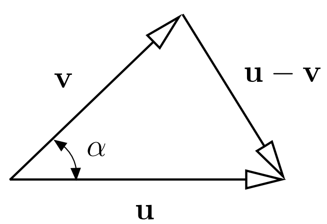
holds, then A, B and C define the "proper EUCLIDean vector space $\mathcal{V}^3$" (physical space).

Square and norm of a vector

$$\mathbf{v}^2 := \mathbf{v} \cdot \mathbf{v} \quad , \quad v = |\mathbf{v}| = \sqrt{\mathbf{v}^2}$$

Rem.: The norm is the value or the positive square root of the vector.

Angle between two vectors



$$\sphericalangle(\mathbf{u}\,;\,\mathbf{v}) =: \alpha$$

Law of cosines

$$
\begin{aligned}
|\mathbf{u}-\mathbf{v}|^2 &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2\,|\mathbf{u}|\,|\mathbf{v}|\,\cos\alpha \\
\longrightarrow \qquad \cos\alpha &= \frac{\mathbf{u}^2 + \mathbf{v}^2 - (\mathbf{u}-\mathbf{v})^2}{2\,|\mathbf{u}|\,|\mathbf{v}|} \\
&= \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}
\end{aligned}
$$

or $$\boxed{\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\,\cos\alpha}$$

Scalar products (inner products) in an orthonormal basis

Scalar product of the basis vectors $\mathbf{e}_i$:

$$
\sphericalangle(\mathbf{e}_i\,;\,\mathbf{e}_k) \quad \begin{cases} 90^\circ & \text{if } i \neq k \quad : \quad \cos 90^\circ = 0 \\ 0^\circ & \text{if } i = k \quad : \quad \cos 0^\circ = 1 \end{cases}
$$

thus

$$
\begin{aligned}
\mathbf{e}_i\cdot\mathbf{e}_k &= |\mathbf{e}_i|\,|\mathbf{e}_k|\,\cos\,\sphericalangle(\mathbf{e}_i\,;\,\mathbf{e}_k) \\
&= \cos\,\sphericalangle(\mathbf{e}_i\,;\,\mathbf{e}_k)
\end{aligned}
$$

It follows with the KRONECKER δ

$$
\boxed{\mathbf{e}_i\cdot\mathbf{e}_k = \delta_{ik} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}}
$$

Scalar product of two vectors:

$$
\begin{aligned}
\mathbf{u}\cdot\mathbf{v} &= (u_i\,\mathbf{e}_i)\cdot(v_k\,\mathbf{e}_k) \\
&= u_i\,v_k\,(\mathbf{e}_i\cdot\mathbf{e}_k) \\
&= u_i\,v_k\,\delta_{ik} \\
&= u_i\,v_i = u_1\,v_1 + u_2\,v_2 + u_3\,v_3
\end{aligned}
$$

D: Vector or cross product (outer product) of vectors

One defines the following vector product

$\mathbf{u}\times\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\,\sin\,\sphericalangle(\mathbf{u}\,;\,\mathbf{v})\,\mathbf{n}$

with $\mathbf{n}$: unit vector $\perp \mathbf{u}\,,\ \mathbf{v}$ (corkscrew rule or right-hand rule, see page 7)

From the above definiton, the following relations can be derived

$$
\begin{aligned}
\mathbf{u}\times\mathbf{v} &= -\mathbf{v}\times\mathbf{u} && : \underline{\text{no}} \text{ commutative law} \\
\mathbf{u}\times(\mathbf{v}+\mathbf{w}) &= \mathbf{u}\times\mathbf{v} + \mathbf{u}\times\mathbf{w} && : \text{distributive law} \\
\alpha\,(\mathbf{u}\times\mathbf{v}) &= (\alpha\,\mathbf{u})\times\mathbf{v} = \mathbf{u}\times(\alpha\,\mathbf{v}) && : \text{associative law}
\end{aligned}
$$

Scalar triple product (parallelepidial product):

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \mathbf{v}\cdot(\mathbf{w}\times\mathbf{u}) = \mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})$$

Arithmetic laws for the vector product (without proof)

$$\begin{aligned} \mathbf{u}\times\mathbf{u} &= \mathbf{0} \\ (\mathbf{u}+\mathbf{v})\times\mathbf{w} &= \mathbf{u}\times\mathbf{w}+\mathbf{v}\times\mathbf{w} \\ \mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) &= \mathbf{v}\cdot(\mathbf{u}\times\mathbf{u}) = 0 \end{aligned}$$

Expansion theorem:

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\,\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\,\mathbf{w}$$

LAGRANGEan identity (Jean Louis Lagrange: 1736-1813):

$$(\mathbf{u}\times\mathbf{v})\cdot(\mathbf{w}\times\mathbf{z}) = (\mathbf{u}\cdot\mathbf{w})\,(\mathbf{v}\cdot\mathbf{z}) - (\mathbf{u}\cdot\mathbf{z})\,(\mathbf{v}\cdot\mathbf{w})$$

Norm of the vector product:

$$|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\,\sin\sphericalangle(\mathbf{u}\,;\,\mathbf{v})$$

Vector product in an orthonormal basis

here: simplified representation in matrix notation

Calculation of

$$\begin{aligned} \mathbf{u} \;=\; \mathbf{v}\times\mathbf{w} &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= (v_2\,w_3 - v_3\,w_2)\,\mathbf{e}_1 - (v_1\,w_3 - v_3\,w_1)\,\mathbf{e}_2 + (v_1\,w_2 - v_2\,w_1)\,\mathbf{e}_3 \end{aligned}$$

Rem.: $\mathbf{u}\perp\mathbf{v},\,\mathbf{w}$; i. e. $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w} = 0$ holds

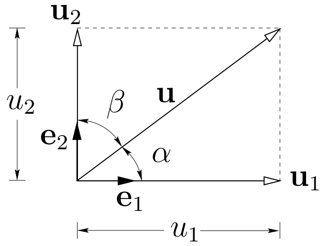
Example:

$$\mathbf{u}\cdot\mathbf{v} = u_i\,v_i = (v_2\,w_3 - v_3\,w_2)\,v_1 - (v_1\,w_3 - v_3\,w_1)\,v_2 + (v_1\,w_2 - v_2\,w_1)\,v_3 = 0 \quad \text{q. e. d.}$$

Remarks on the products between vectors

• on the scalar product

Decomposition of a vector (example: in 2-d):



$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$

with $\mathbf{u}_1 = u_1\,\mathbf{e}_1$ and $\mathbf{u}_2 = u_2\,\mathbf{e}_2$

$\mathbf{u}_1\,,\ \mathbf{u}_2$: vector components

$u_1\,,\ u_2$: coefficients of the vector components

Projection of $\mathbf{u}$ on the directions of $\mathbf{e}_i$:

$$\boxed{u_i \;=\; \mathbf{u} \cdot \mathbf{e}_i}$$

Verification of the projection law:

$$\begin{aligned} \mathbf{u} \cdot \mathbf{e}_i &= (u_k\, \mathbf{e}_k) \cdot \mathbf{e}_i \\ &= u_k\, \delta_{ki} \;=\; u_i \quad \text{q. e. d.} \end{aligned}$$

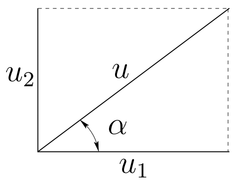
Calculation of the projections:

$$\begin{aligned} u_1 &= |\mathbf{u}|\, |\mathbf{e}_1| \cos\alpha \\ &= |\mathbf{u}| \cos\alpha \;=\; u \cos\alpha \end{aligned}$$

with $u \;=\; |\mathbf{u}|$

$$\begin{aligned} u_2 &= u \cos\beta \\ &= u \cos(90^\circ - \alpha) \;=\; u \sin\alpha \end{aligned}$$

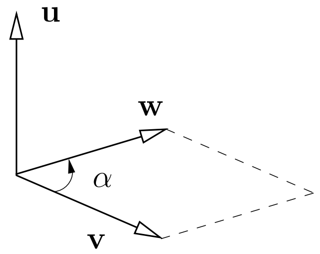
Note: For the values of the vector components, the following relations hold



$$\begin{aligned} u_1 &= u \cos\alpha \\ u_2 &= u \sin\alpha \end{aligned}$$

- **on the vector product**

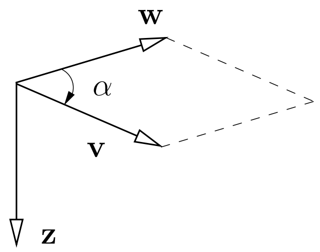
Orientation of the vector $\mathbf{u} \;=\; \mathbf{v} \times \mathbf{w}$:



(1) $\mathbf{u} \perp \mathbf{v}\,,\ \mathbf{w}$

(2) corkscrew rule (right-hand rule)

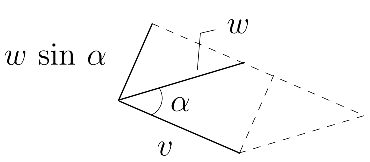
It is obvious that



$$\mathbf{z} \;=\; \mathbf{w} \times \mathbf{v}$$

$$\longrightarrow \quad \mathbf{v} \times \mathbf{w} \;=\; -\,\mathbf{w} \times \mathbf{v}$$

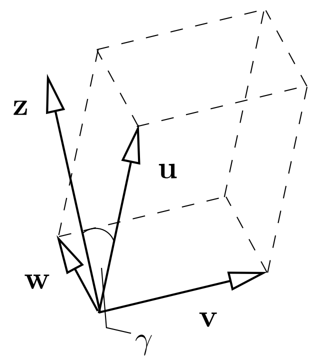
Value of the vector product:



$$\begin{aligned} |\mathbf{v} \times \mathbf{w}| &= |\mathbf{v}|\, |\mathbf{w}| \sin\alpha \\ &= v\,(w \sin\alpha) \end{aligned}$$

Note:	The vector $\mathbf{v} \times \mathbf{w}$ is perpendicular to $\mathbf{v}$ and $\mathbf{w}$ (corkscrew orientation); its value corresponds to the area spanned by $\mathbf{v}$ and $\mathbf{w}$.

Scalar triple product (parallelepidial product):



$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \;=:\; [\,\mathbf{u}\,\mathbf{v}\,\mathbf{w}\,]$$

with $\mathbf{z} = \mathbf{v} \times \mathbf{w}$

follows
$$\mathbf{u} \cdot \mathbf{z} = |\mathbf{u}|\,|\mathbf{z}|\,\cos\gamma = z\,(u\,\cos\gamma)$$

with $(u\,\cos\gamma)$: projection of $\mathbf{u}$ on the direction of $\mathbf{z}$

Rem.: The parallelepidial product yields the volume of the parallelepiped spanned by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

Remark:	The preceding and the following relations are valid with respect to an arbitrary basis system. For simplicity, the following material is restricted to the orthonormal basis, whenever a basis notation occurs. Concerning a more general basis representation, cf., e. g., DE BOER, R.: Vektor- und Tensorrechnung für Ingenieure. Springer-Verlag, Berlin 1982.

2 Fundamentals of tensor calculus

Rem.: The following statements are related to the proper EUKLIDian vector space $\mathcal{V}^3$ and the corresponding dyadic product space $\mathcal{V}^3 \otimes \mathcal{V}^3 \otimes \cdots \otimes \mathcal{V}^3$ (n times) of n-th order.

2.1 Introduction of the tensor concept

(a) Tensor concept and linear mapping

Definition: A 2nd order (2nd rank) tensor $\mathbf{T}$ is a linear mapping which transforms a vector $\mathbf{u}$ uniquely in a vector $\mathbf{w}$:

$$\mathbf{w} = \mathbf{T}\,\mathbf{u}$$

therein: $\left\{ \begin{array}{ll} \mathbf{u},\,\mathbf{w} \in \mathcal{V}^3 & ; \quad \mathbf{T} \in \mathcal{L}(\mathcal{V}^3,\,\mathcal{V}^3) \\ \mathcal{L}(\mathcal{V}^3,\,\mathcal{V}^3) & : \quad \text{set of all 2nd order tensors or linear mappings of vectors, respectively} \end{array} \right.$

(b) Tensor concept and dyadic product space

Definition: There is a "simple tensor" $(\mathbf{a} \otimes \mathbf{b})$ with the property

$$(\mathbf{a} \otimes \mathbf{b})\,\mathbf{c} =: (\mathbf{b} \cdot \mathbf{c})\,\mathbf{a}$$

therein: $\left\{ \begin{array}{l} \mathbf{a} \otimes \mathbf{b} \;\in\; \mathcal{V}^3 \otimes \mathcal{V}^3 \text{ (dyadic product space)} \\ \otimes \;:\; \text{dyadic product (binary operator of } \mathcal{V}^3 \otimes \mathcal{V}^3) \end{array} \right.$

It follows directly that

$$\mathbf{a} \otimes \mathbf{b} \;\in\; \mathcal{L}(\mathcal{V}^3,\,\mathcal{V}^3) \quad \longrightarrow \quad \mathcal{V}^3 \otimes \mathcal{V}^3 \subset \mathcal{L}(\mathcal{V}^3,\,\mathcal{V}^3)$$

Rem.: $(\mathbf{a} \otimes \mathbf{b})$ maps a vector $\mathbf{c}$ onto a vector $\mathbf{d} \;=\; (\mathbf{b} \cdot \mathbf{c})\,\mathbf{a}$.

Basis notation of a simple tensor:

$$\mathbf{A} := \mathbf{a} \otimes \mathbf{b} = (a_i\,\mathbf{e}_i) \otimes (b_k\,\mathbf{e}_k) = a_i\,b_k\,(\mathbf{e}_i \otimes \mathbf{e}_k)$$

with $\left\{ \begin{array}{ll} a_i\,b_k & : \text{coefficients of the tensor components} \\ \mathbf{e}_i \otimes \mathbf{e}_k & : \text{tensor basis} \end{array} \right.$

Tensors $\mathbf{A} \in \mathcal{V}^3 \otimes \mathcal{V}^3$ have 9 independent components (and directions);
e. g. $a_1\,b_3\,(\mathbf{e}_1 \otimes \mathbf{e}_3)$ etc.

Introduction of arbitrary tensors $\mathbf{T} \in \mathcal{V}^3 \otimes \mathcal{V}^3$:

$$\mathbf{T} = \quad t_{ik}\,(\mathbf{e}_i \otimes \mathbf{e}_k)$$

$$\text{with} \quad t_{ik} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} \quad : \quad \left\{ \begin{array}{l} \text{matrix of coefficients of } \mathbf{T} \text{ with} \\ \text{9 independent quantities} \end{array} \right.$$

2.2 Basic rules of tensor algebra

Requirement: $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, ...\} \in \mathcal{V}^3 \otimes \mathcal{V}^3$.

(a) Tensor addition

$$\begin{aligned} \mathbf{A} + \mathbf{B} &= \mathbf{B} + \mathbf{A} && : \text{ commutative law} \\ \mathbf{A} + (\mathbf{B} + \mathbf{C}) &= (\mathbf{A} + \mathbf{B}) + \mathbf{C} && : \text{ associative law} \\ \mathbf{A} + \mathbf{0} &= \mathbf{A} && : \ \mathbf{0} \quad : \text{ identical element} \\ \mathbf{A} + (-\mathbf{A}) &= \mathbf{0} && : -\mathbf{A} \quad : \text{ inverse element} \end{aligned}$$

Tensor addition with respect to an orthonormal tensor basis:

$$\mathbf{A} = a_{ik}\,(\mathbf{e}_i \otimes \mathbf{e}_k), \qquad \mathbf{B} = b_{ik}\,(\mathbf{e}_i \otimes \mathbf{e}_k)$$

$$\longrightarrow \quad \mathbf{C} = \mathbf{A} + \mathbf{B} = \underbrace{(a_{ik} + b_{ik})}_{c_{ik}}(\mathbf{e}_i \otimes \mathbf{e}_k)$$

Rem.: A tensor addition carried out as an addition of the tensor coefficients requires that both tensors have the same tensor basis.

(b) Multiplication of tensors by a scalar

$$\begin{aligned} 1\,\mathbf{A} &= \mathbf{A} && : 1 \ : \text{ identical element} \\ \alpha\,(\beta\,\mathbf{A}) &= (\alpha\,\beta)\,\mathbf{A} && : \text{associative law} \\ (\alpha + \beta)\,\mathbf{A} &= \alpha\,\mathbf{A} + \beta\,\mathbf{A} && : \text{distributive law (with respect to the addition of scalars)} \\ \alpha\,(\mathbf{A} + \mathbf{B}) &= \alpha\,\mathbf{A} + \alpha\,\mathbf{B} && : \text{distributive law (with respect to the addition of tensors)} \\ \alpha\,\mathbf{A} &= \mathbf{A}\,\alpha && : \text{commutative law} \end{aligned}$$

(c) Linear mapping between tensor and vector

The following definitions make use of the linear mapping (cf. 2.1)

$$\mathbf{w} = \mathbf{T}\,\mathbf{u}$$

Rem.: In the literature, the multiplication of a vector by a tensor is also called "contraction".

The following relations hold:

$$\begin{aligned} \mathbf{A}\,(\mathbf{u}+\mathbf{v}) &= \mathbf{A}\,\mathbf{u}+\mathbf{A}\,\mathbf{v} && : \text{distributive law} \\ \mathbf{A}\,(\alpha\,\mathbf{u}) &= \alpha(\mathbf{A}\,\mathbf{u}) && : \text{associative law} \\ (\mathbf{A}+\mathbf{B})\,\mathbf{u} &= \mathbf{A}\,\mathbf{u}+\mathbf{B}\,\mathbf{u} && : \text{distributive law} \\ (\alpha\,\mathbf{A})\,\mathbf{u} &= \alpha\,(\mathbf{A}\,\mathbf{u}) && : \text{associative law} \\ \mathbf{0}\,\mathbf{u} &= \mathbf{0} && : \ \mathbf{0}\ : \text{zero element of the linear mapping} \\ \mathbf{I}\,\mathbf{u} &= \mathbf{u} && : \ \mathbf{I}\ : \text{identity element of the linear mapping} \end{aligned}$$

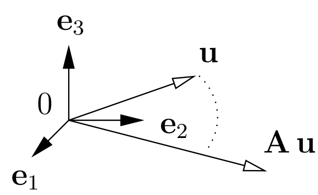
Linear mapping in basis notation:

$$\begin{aligned} \mathbf{A} &= a_{ik}\,(\mathbf{e}_i\otimes\mathbf{e}_k)\,, \qquad \mathbf{u}=u_i\,\mathbf{e}_i \\ \mathbf{A}\,\mathbf{u} &= (a_{ik}\,\mathbf{e}_i\otimes\mathbf{e}_k)\,(u_j\mathbf{e}_j) \;=\; a_{ik}\,u_j\,(\mathbf{e}_i\otimes\mathbf{e}_k)\,\mathbf{e}_j \end{aligned}$$

One obtains

$$\mathbf{w} = \mathbf{A}\,\mathbf{u} = a_{ik}\,u_j\,\delta_{kj}\,\mathbf{e}_i = \underbrace{a_{ik}\,u_k}_{w_i}\,\mathbf{e}_i \quad \text{mit} \quad \begin{cases} \text{i} & : \text{ free index (basis index)} \\ \text{k} & : \text{ silent index (double index of } w_i) \end{cases}$$

Rem.: In general, a linear mapping $\mathbf{A}$ causes both a rotation **and** a stretch of a vector $\mathbf{u}$.



Identity tensor $\mathbf{I}\in\mathcal{V}^3\otimes\mathcal{V}^3$:

$$\mathbf{I} = \delta_{ik}\,\mathbf{e}_i\otimes\mathbf{e}_k = \mathbf{e}_i\otimes\mathbf{e}_i$$

Proof of the defining property:

$$\mathbf{u} = \mathbf{I}\,\mathbf{u} = (\mathbf{e}_i\otimes\mathbf{e}_i)\,u_j\,\mathbf{e}_j = u_j\,(\mathbf{e}_i\otimes\mathbf{e}_i)\,\mathbf{e}_j = u_j\,\delta_{ij}\,\mathbf{e}_i = u_i\,\mathbf{e}_i \quad \text{q. e. d.}$$

Rem.: Tensors built from basis vectors are called fundamental tensors, i. e.

$$\boxed{\mathbf{I}\in\mathcal{V}^3\otimes\mathcal{V}^3 \text{ is the fundamental tensor of 2nd order.}}$$

(d) Scalar product of tensors (inner product)

The following relations hold:

$$\begin{aligned} \mathbf{A}\cdot\mathbf{B} &= \mathbf{B}\cdot\mathbf{A} && : \text{commutative law} \\ \mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) &= \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C} && : \text{distributive law} \\ (\alpha\,\mathbf{A})\cdot\mathbf{B} &= \mathbf{A}\cdot(\alpha\,\mathbf{B}) = \alpha\,(\mathbf{A}\cdot\mathbf{B}) && : \text{associative law} \\ \mathbf{A}\cdot\mathbf{B} &= 0 \quad \forall\,\mathbf{A}\,,\ \text{if } \mathbf{B}\equiv\mathbf{0} \\ &\longrightarrow\ \mathbf{A}\cdot\mathbf{A} > 0 \text{ for } \mathbf{A}\neq\mathbf{0} \end{aligned}$$

Scalar product of $\mathbf{A}$ with a simple tensor $\mathbf{a} \otimes \mathbf{b} \in \mathcal{V}^3 \otimes \mathcal{V}^3$:

$$\mathbf{A} \cdot (\mathbf{a} \otimes \mathbf{b}) = \mathbf{a} \cdot \mathbf{A}\,\mathbf{b}$$

Scalar product of A and B in basis notation:

$$\begin{aligned} \mathbf{A} &= a_{ik}\,(\mathbf{e}_i \otimes \mathbf{e}_k),\ \mathbf{B} = b_{ik}\,(\mathbf{e}_i \otimes \mathbf{e}_k) \\ \alpha &= \mathbf{A} \cdot \mathbf{B} = a_{ik}\,(\mathbf{e}_i \otimes \mathbf{e}_k) \cdot b_{st}(\mathbf{e}_s \otimes \mathbf{e}_t) = a_{ik}\,b_{st}(\mathbf{e}_i \otimes \mathbf{e}_k) \cdot (\mathbf{e}_s \otimes \mathbf{e}_t) \end{aligned}$$

One obtains

$$\alpha = a_{ik}\,b_{st}\,\delta_{is}\,\delta_{kt} = a_{ik}\,b_{ik}$$

Rem.: The result of the scalar product is a scalar.

(e) Tensor product of tensors

Definition: The tensor product of tensors yields

$$(\mathbf{AB})\,\mathbf{v} = \mathbf{A}\,(\mathbf{B}\,\mathbf{v})$$

Rem.: With this definition, the tensor product of tensors is directly linked to the linear mapping (cf. 2.1 (a)).

The following relations hold:

$$\begin{aligned} (\mathbf{A}\,\mathbf{B})\,\mathbf{C} &= \mathbf{A}\,(\mathbf{B}\,\mathbf{C}) && : \text{associative law} \\ \mathbf{A}\,(\mathbf{B}+\mathbf{C}) &= \mathbf{A}\,\mathbf{B} + \mathbf{A}\,\mathbf{C} && : \text{distributive law} \\ (\mathbf{A}+\mathbf{B})\,\mathbf{C} &= \mathbf{A}\,\mathbf{C} + \mathbf{B}\,\mathbf{C} && : \text{distributive law} \\ \alpha\,(\mathbf{A}\,\mathbf{B}) &= (\alpha\,\mathbf{A})\,\mathbf{B} = \mathbf{A}\,(\alpha\,\mathbf{B}) && : \text{associative law} \\ \mathbf{I}\,\mathbf{T} &= \mathbf{T}\,\mathbf{I} = \mathbf{T} && : \mathbf{I}\ :\ \text{identity element} \\ \mathbf{0}\,\mathbf{T} &= \mathbf{T}\,\mathbf{0} = \mathbf{0} && : \mathbf{0}\ :\ \text{zero element} \end{aligned}$$

Rem.: In general, the commutative law is not valid, i. e. $\mathbf{A}\,\mathbf{B} \neq \mathbf{B}\,\mathbf{A}$.

Tensor product of simple tensors:

$$\mathbf{A} = \mathbf{a} \otimes \mathbf{b}\,, \quad \mathbf{B} = \mathbf{c} \otimes \mathbf{d}$$

It follows with the above definition

$$\begin{aligned} (\mathbf{A}\,\mathbf{B})\,\mathbf{v} &= \mathbf{A}\,(\mathbf{B}\,\mathbf{v}) \\ \longrightarrow [\,(\mathbf{a} \otimes \mathbf{b})\,(\mathbf{c} \otimes \mathbf{d})\,]\,\mathbf{v} &= (\mathbf{a} \otimes \mathbf{b})\,[\,(\mathbf{c} \otimes \mathbf{d})\,\mathbf{v}\,] \\ &= (\mathbf{a} \otimes \mathbf{b})\,(\mathbf{d} \cdot \mathbf{v})\,\mathbf{c} \\ &= (\mathbf{b} \cdot \mathbf{c})\,(\mathbf{d} \cdot \mathbf{v})\,\mathbf{a} \\ &= [\,(\mathbf{b} \cdot \mathbf{c})\,(\mathbf{a} \otimes \mathbf{d})\,]\,\mathbf{v} \end{aligned}$$

Consequence:

$$(\mathbf{a} \otimes \mathbf{b})\,(\mathbf{c} \otimes \mathbf{d}) \;=\; (\mathbf{b} \cdot \mathbf{c})\,\mathbf{a} \otimes \mathbf{d}$$

Tensor product in basis notation:

$$\begin{aligned}
\mathbf{A}\,\mathbf{B} &= a_{ik}\,(\mathbf{e}_i \otimes \mathbf{e}_k)\,b_{st}(\mathbf{e}_s \otimes \mathbf{e}_t) \\
&= a_{ik}\,b_{st}\,(\mathbf{e}_i \otimes \mathbf{e}_k)\,(\mathbf{e}_s \otimes \mathbf{e}_t) \\
&= a_{ik}\,b_{st}\,\delta_{ks}(\mathbf{e}_i \otimes \mathbf{e}_t) \\
&= a_{ik}\,b_{kt}\,(\mathbf{e}_i \otimes \mathbf{e}_t)
\end{aligned}$$

Rem.: The result of a tensor product is a tensor.

2.3 Specific tensors and operations

(a) Transposed tensor

Definition: The transposed tensor $\mathbf{A}^T$ belonging to $\mathbf{A}$ exhibits the property

$$\mathbf{w} \cdot (\mathbf{A}\,\mathbf{u}) = (\mathbf{A}^T\,\mathbf{w}) \cdot \mathbf{u}$$

The following relations hold:

$$\begin{aligned}
(\mathbf{A} + \mathbf{B})^T &= \mathbf{A}^T + \mathbf{B}^T \\
(\alpha\,\mathbf{A})^T &= \alpha\,\mathbf{A}^T \\
(\mathbf{A}\,\mathbf{B})^T &= \mathbf{B}^T\,\mathbf{A}^T
\end{aligned}$$

Transposition of a simple tensor $\mathbf{a} \otimes \mathbf{b}$:

It follows with the above definition

$$\begin{aligned}
\mathbf{w} \cdot\, (\mathbf{a} \otimes \mathbf{b})\,\mathbf{u} &= \mathbf{w} \cdot (\mathbf{b} \cdot \mathbf{u})\,\mathbf{a} \\
&= (\mathbf{w} \cdot \mathbf{a})\,(\mathbf{b} \cdot \mathbf{u}) \\
&= (\mathbf{b} \otimes \mathbf{a})\,\mathbf{w} \cdot \mathbf{u} \\
\longrightarrow (\mathbf{a} \otimes \mathbf{b})^T &= \mathbf{b} \otimes \mathbf{a}
\end{aligned}$$

Transposed tensor in basis notation:

$$\begin{aligned}
\mathbf{A} &= a_{ik}\,(\mathbf{e}_i \otimes \mathbf{e}_k) \\
\longrightarrow\; \mathbf{A}^T &= a_{ik}\,(\mathbf{e}_k \otimes \mathbf{e}_i) \\
&= a_{ki}\,(\mathbf{e}_i \otimes \mathbf{e}_k) \quad \text{: renaming the indices}
\end{aligned}$$

Note: The transposition of a tensor $\mathbf{A} \in \mathcal{V}^3 \otimes \mathcal{V}^3$ can be carried out by an exchange of the tensor basis **or** by an exchange of the subscripts of the tensor coefficients.

(b) Symmetric and skew-symmetric tensor

Definition: A tensor $\mathbf{A} \in \mathcal{V}^3 \otimes \mathcal{V}^3$ is symmetric, if

$$\mathbf{A} = \mathbf{A}^T$$

and skew-symmetric (antimetric), if

$$\mathbf{A} = -\mathbf{A}^T$$

Symmetric and skew-symmetric parts of an arbitrary tensor $\mathbf{A} \in \mathcal{V}^3 \otimes \mathcal{V}^3$:

$$\begin{aligned} \operatorname{sym} \mathbf{A} &= \tfrac{1}{2}(\mathbf{A} + \mathbf{A}^T) \\ \operatorname{skw} \mathbf{A} &= \tfrac{1}{2}(\mathbf{A} - \mathbf{A}^T) \\ \longrightarrow \mathbf{A} &= \operatorname{sym} \mathbf{A} + \operatorname{skw} \mathbf{A} \end{aligned}$$

Properties of symmetric and skew-symmetric tensors:

$$\begin{aligned} \mathbf{w} \cdot (\operatorname{sym} \mathbf{A})\,\mathbf{v} &= (\operatorname{sym} \mathbf{A})\,\mathbf{w} \cdot \mathbf{v} \\ \mathbf{v} \cdot (\operatorname{skw} \mathbf{A})\,\mathbf{v} &= -(\operatorname{skw} \mathbf{A})\,\mathbf{v} \cdot \mathbf{v} = 0 \end{aligned}$$

Positive definite symmetric tensors:

- $\operatorname{sym} \mathbf{A}$ is positive definite, if $\quad \operatorname{sym} \mathbf{A} \cdot (\mathbf{v} \otimes \mathbf{v}) = \mathbf{v} \cdot (\operatorname{sym} \mathbf{A})\,\mathbf{v} > 0$
- $\operatorname{sym} \mathbf{A}$ is positive semi-definite, if $\quad \operatorname{sym} \mathbf{A} \cdot (\mathbf{v} \otimes \mathbf{v}) = \mathbf{v} \cdot (\operatorname{sym} \mathbf{A})\,\mathbf{v} \geq 0$

(c) Inverse tensor

Definition: If $\mathbf{A}^{-1}$ inverse to $\mathbf{A}$ exists, it exhibits the property

$$\mathbf{v} = \mathbf{A}\,\mathbf{w} \quad \longleftrightarrow \quad \mathbf{w} = \mathbf{A}^{-1}\,\mathbf{v}$$

The following relations hold:

$$\begin{aligned} \mathbf{A}\,\mathbf{A}^{-1} &= \mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I} \\ (\mathbf{A}^{-1})^T &= (\mathbf{A}^T)^{-1} =: \mathbf{A}^{T-1} \\ (\mathbf{A}\,\mathbf{B})^{-1} &= \mathbf{B}^{-1}\,\mathbf{A}^{-1} \end{aligned}$$

Rem.: The computation of the inverse tensor in basis notation is carried out by introducing the "double cross product" (outer tensor product of tensors), cf. 2.8.

(d) Orthogonal tensor

Definition: An orthogonal tensor $\mathbf{Q} \in \mathcal{V}^3 \otimes \mathcal{V}^3$ exhibits the property

$$\mathbf{Q}^{-1} = \mathbf{Q}^T \quad \longleftrightarrow \quad \mathbf{Q}\,\mathbf{Q}^T = \mathbf{I}$$

$$\text{Additionally} \begin{cases} (\det \mathbf{Q})^2 = 1 & : \text{orthogonality} \\ \det \mathbf{Q} = 1 & : \text{proper orthogonality} \end{cases}$$

Rem.: The computation of the determinant of 2nd order tensors is defined with the aid of the double cross product, cf. 2.8.

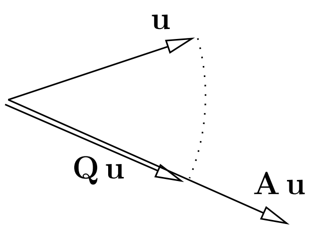
Properties of orthogonal tensors:

$$\mathbf{Q}\,\mathbf{v} \cdot \mathbf{Q}\,\mathbf{w} = \mathbf{Q}^T\,\mathbf{Q}\,\mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w}$$

$$\longrightarrow \quad \mathbf{Q}\,\mathbf{u} \cdot \mathbf{Q}\,\mathbf{u} = \mathbf{u} \cdot \mathbf{u}$$

Rem.: Linear mapping with $\mathbf{Q}$ preserves the norm of the respective vector.

Illustration:



in general: linear mapping with $\mathbf{A} \in \mathcal{V}^3 \otimes \mathcal{V}^3$ causes a rotation **and** a stretch

in special: linear mapping with $\mathbf{Q} \in \mathcal{V}^3 \otimes \mathcal{V}^3$ causes **only** a rotation

(e) Trace of a tensor

Definition: The trace $\operatorname{tr}\mathbf{A}$ of a tensor $\mathbf{A} \in \mathcal{V}^3 \otimes \mathcal{V}^3$ is the scalar product

$$\operatorname{tr}\mathbf{A} = \mathbf{A} \cdot \mathbf{I}$$

The following relations hold:

$$\begin{aligned}
\operatorname{tr}(\alpha\,\mathbf{A}) &= \alpha \operatorname{tr}\mathbf{A} \\
\operatorname{tr}(\mathbf{a} \otimes \mathbf{b}) &= \mathbf{a} \cdot \mathbf{b} \\
\operatorname{tr}\mathbf{A}^T &= \operatorname{tr}\mathbf{A} \\
\operatorname{tr}(\mathbf{A}\,\mathbf{B}) &= \operatorname{tr}(\mathbf{B}\mathbf{A}) \\
\longrightarrow \quad (\mathbf{A}\,\mathbf{B}) \cdot \mathbf{I} &= \mathbf{B} \cdot \mathbf{A}^T = \mathbf{B}^T \cdot \mathbf{A} \\
\operatorname{tr}(\mathbf{A}\,\mathbf{B}\,\mathbf{C}) &= \operatorname{tr}(\mathbf{B}\,\mathbf{C}\,\mathbf{A}) = \operatorname{tr}(\mathbf{C}\,\mathbf{A}\,\mathbf{B})
\end{aligned}$$

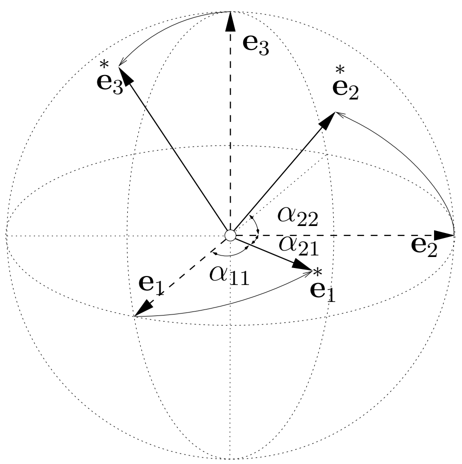
2.4 Change of the basis

Rem.: The goal is to find a relation between vectors and tensors which belong to different basis systems.

here: Restriction to orthonormal basis systems which are rotated against each other.

(A) Rotation of the basis system

Illustration:



$\{\, 0, \mathbf{e}_i \}$: basis system

$\{\, 0, \overset{*}{\mathbf{e}}_i \}$: rotated basis system

$\{\, \alpha_{ik} \}$: angle between the basis vectors $\mathbf{e}_i$ and $\overset{*}{\mathbf{e}}_k$

Development of the transformation tensor:

The following relations hold:

$$\overset{*}{\mathbf{e}}_i = \mathbf{I}\,\overset{*}{\mathbf{e}}_i \quad \text{and} \quad \mathbf{I} = \mathbf{e}_j \otimes \mathbf{e}_j$$

Thus,

$$\overset{*}{\mathbf{e}}_i = (\mathbf{e}_j \otimes \mathbf{e}_j)\,\overset{*}{\mathbf{e}}_i = (\mathbf{e}_j \cdot \overset{*}{\mathbf{e}}_i)\,\mathbf{e}_j$$

using $\overset{*}{\mathbf{e}}_i = \delta_{ik}\,\overset{*}{\mathbf{e}}_k$ leads to

$$\overset{*}{\mathbf{e}}_i = (\mathbf{e}_j \cdot \delta_{ik}\,\overset{*}{\mathbf{e}}_k)\,\mathbf{e}_j = (\mathbf{e}_j \cdot \overset{*}{\mathbf{e}}_k)\,(\mathbf{e}_i \cdot \mathbf{e}_k)\,\mathbf{e}_j$$

one obtains

$$\overset{*}{\mathbf{e}}_i = (\mathbf{e}_j \cdot \overset{*}{\mathbf{e}}_k)\,(\mathbf{e}_j \otimes \mathbf{e}_k)\mathbf{e}_i =:\ \mathbf{R}\,\mathbf{e}_i \quad \text{with} \quad \mathbf{R} = (\mathbf{e}_j \cdot \overset{*}{\mathbf{e}}_k)\,\mathbf{e}_j \otimes \mathbf{e}_k$$

Rem.: $\mathbf{R}$ is the transformation tensor which transforms the basis vectors $\mathbf{e}_i$ into the basis vectors $\overset{*}{\mathbf{e}}_i$.

Coefficient matrix R_{jk}:

$$R_{jk} = \mathbf{e}_j \cdot \overset{*}{\mathbf{e}}_k = |\mathbf{e}_j|\,|\overset{*}{\mathbf{e}}_k|\,\cos\sphericalangle\,(\mathbf{e}_j; \overset{*}{\mathbf{e}}_k) = \cos\alpha_{jk} \quad \text{with}\ |\mathbf{e}_j| = |\overset{*}{\mathbf{e}}_k| = 1$$

Rem.: R_{jk} contains the 9 cosines of the angles between the directions of the basis vectors $\mathbf{e}_j$ and $\overset{*}{\mathbf{e}}_k$.

Orthogonality of the transformation tensor:

Rem.: By $\mathbf{R}$, the basis vectors $\mathbf{e}_i$ are only rotated towards $\overset{*}{\mathbf{e}}_i$, thus, $\mathbf{R}$ is an orthogonal tensor.

Orthogonality condition:

$$\begin{aligned} \mathbf{R}\,\mathbf{R}^T \overset{!}{=} \mathbf{I} \; &= R_{jk}(\mathbf{e}_j \otimes \mathbf{e}_k)\, R_{pn}\,(\mathbf{e}_n \otimes \mathbf{e}_p) = R_{jk}\, R_{pn}\, \delta_{kn}\, \mathbf{e}_j \otimes \mathbf{e}_p \\ &= R_{jk}\, R_{pk}\,(\mathbf{e}_j \otimes \mathbf{e}_p) \end{aligned}$$

It follows with $\mathbf{I} = \delta_{jp}\,(\mathbf{e}_j \otimes \mathbf{e}_p)$ by comparison of coefficients

$$\boxed{R_{jk}\, R_{pk} = \delta_{jp}} \qquad (*)$$

Rem.: $(*)$ contains 6 constraints for the 9 cosines ($\mathbf{R}\,\mathbf{R}^T = \mathrm{sym}\,(\mathbf{R}\,\mathbf{R}^T)$), i. e. only 3 of 9 trigonometrical functions are independent. Thus, the rotation of the basis system is defined by 3 angles.

(B) Introduction of "CARDANO angles"

Idea: Rotation around 3 axes which are given by the basis directions $\mathbf{e}_i$. This procedure was firstly investigated by Girolamo Cardano (1501-1576).

Procedure: The rotation of the basis system is carried out by 3 independent rotations around the axes $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. Each rotation is expressed by a transformation tensor $\mathbf{R}_i$ (i = 1, 2, 3).

Rotation of $\mathbf{e}_i$ around $\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1$:

$$\overset{*}{\mathbf{e}}_i = \{\mathbf{R}_1\,[\mathbf{R}_2\,(\mathbf{R}_3\,\mathbf{e}_i)]\} = \overset{*}{\mathbf{R}}\,\mathbf{e}_i \qquad \text{with} \qquad \overset{*}{\mathbf{R}} = \mathbf{R}_1\,\mathbf{R}_2\,\mathbf{R}_3$$

Rotation of $\mathbf{e}_i$ around $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$:

$$\bar{\mathbf{e}}_i = \{\mathbf{R}_3\,[\mathbf{R}_2\,(\mathbf{R}_1\,\mathbf{e}_i)]\} = \bar{\mathbf{R}}\,\mathbf{e}_i \qquad \text{with} \qquad \bar{\mathbf{R}} = \mathbf{R}_3\,\mathbf{R}_2\,\mathbf{R}_1$$

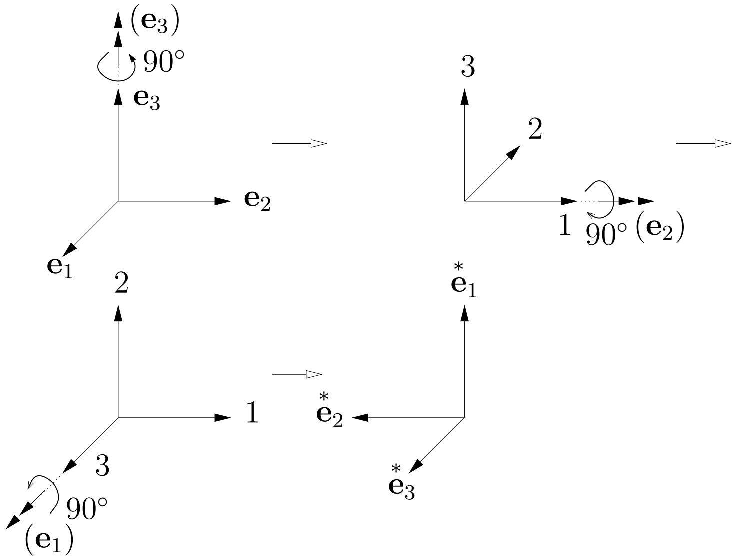
Obviously,

$$\overset{*}{\mathbf{R}} \neq \bar{\mathbf{R}} \quad \longrightarrow \quad \overset{*}{\mathbf{e}} \neq \bar{\mathbf{e}}_i$$

Rem.: The result of the orthogonal transformation depends on the sequence of the rotations.

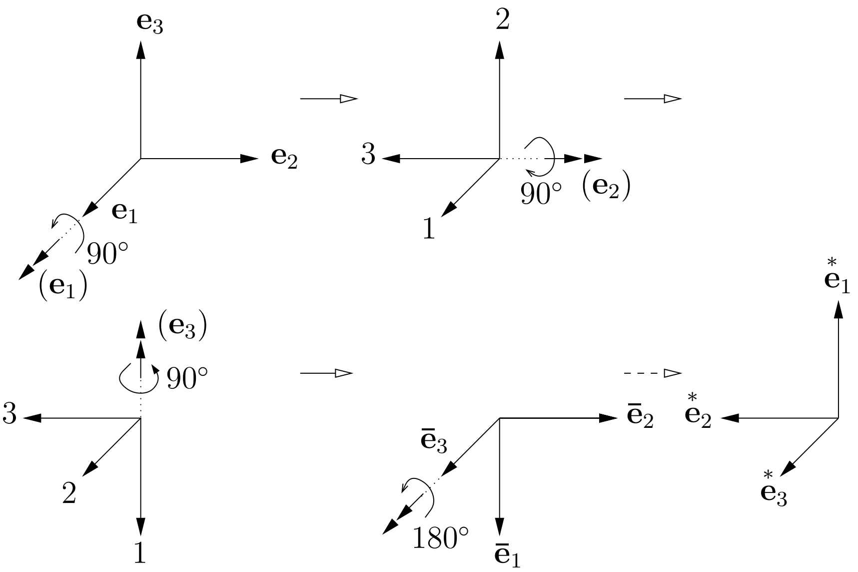
Illustration:

(a) Rotation around $\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1$ (e. g. each about 90°)



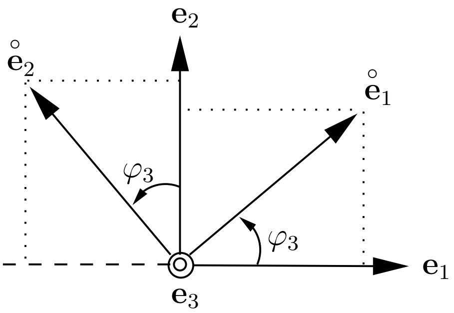
(b) Rotation around $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ (e. g. each about 90°)

with



Definition of the orthogonal rotation tensors $\mathbf{R}_i$

(a) Rotation around the $\mathbf{e}_3$-axis



The following relations hold:

$$\mathring{\mathbf{e}}_1 = \cos\varphi_3\,\mathbf{e}_1 + \sin\varphi_3\,\mathbf{e}_2$$

$$\mathring{\mathbf{e}}_2 = -\sin\varphi_3\,\mathbf{e}_1 + \cos\varphi_3\,\mathbf{e}_2$$

$$\mathring{\mathbf{e}}_3 = \mathbf{e}_3$$

In general,

$$\overset{\circ}{\mathbf{e}}_i = \mathbf{R}_3\,\mathbf{e}_i = R_{3jk}\,(\mathbf{e}_j \otimes \mathbf{e}_k)\,\mathbf{e}_i = R_{3jk}\,\delta_{ki}\,\mathbf{e}_j = R_{3ji}\,\mathbf{e}_j$$

Thus, by comparison of coefficients

$$\mathbf{R}_3 = R_{3ji}\,(\mathbf{e}_j \otimes \mathbf{e}_i) \quad \text{with} \quad R_{3ji} = \begin{bmatrix} \cos\varphi_3 & -\sin\varphi_3 & 0 \\ \sin\varphi_3 & \cos\varphi_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>(b) Rotation around the $\mathbf{e}_2$- and $\mathbf{e}_1$-axis</u>

Analogously,

$$\mathbf{R}_2 = R_{2ji}\,(\mathbf{e}_j \otimes \mathbf{e}_i) \qquad \text{with} \quad R_{2ji} = \begin{bmatrix} \cos\varphi_2 & 0 & \sin\varphi_2 \\ 0 & 1 & 0 \\ -\sin\varphi_2 & 0 & \cos\varphi_2 \end{bmatrix}$$

$$\mathbf{R}_1 = R_{1ji}\,(\mathbf{e}_j \otimes \mathbf{e}_i) \qquad \text{with} \quad R_{1ji} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_1 & -\sin\varphi_1 \\ 0 & \sin\varphi_1 & \cos\varphi_1 \end{bmatrix}$$

Rem.: The rotation tensor $\mathbf{R}$ can be composed of single rotations under consideration of the rotation sequence.

<u>(c) Definition of the total rotation $\mathbf{R}$</u>

(c_1) it follows from rotation of $\mathbf{e}_i$ around $\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1$ that

$$\begin{aligned} \mathbf{R} \longrightarrow \overset{*}{\mathbf{R}} &= \mathbf{R}_1\,\mathbf{R}_2\,\mathbf{R}_3 \\ &= R_{1ij}\,(\mathbf{e}_i \otimes \mathbf{e}_j)\,R_{2no}\,(\mathbf{e}_n \otimes \mathbf{e}_o)\,R_{3pq}\,(\mathbf{e}_p \otimes \mathbf{e}_q) \\ &= R_{1ij}\,R_{2no}\,R_{3pq}\,\delta_{jn}\,\delta_{op}\,(\mathbf{e}_i \otimes \mathbf{e}_q) \\ &= \underbrace{R_{1ij}\,R_{2jo}\,R_{3oq}}_{\overset{*}{R}_{iq}}\,(\mathbf{e}_i \otimes \mathbf{e}_q) \end{aligned}$$

with

$$\overset{*}{R}_{iq} = \begin{bmatrix} \cos\varphi_2\,\cos\varphi_3 & -\cos\varphi_2\,\sin\varphi_3 & \sin\varphi_2 \\ \sin\varphi_1\,\sin\varphi_2\,\cos\varphi_3 + \cos\varphi_1\,\sin\varphi_3 & -\sin\varphi_1\,\sin\varphi_2\,\sin\varphi_3 + \cos\varphi_1\,\cos\varphi_3 & -\sin\varphi_1\,\cos\varphi_2 \\ -\cos\varphi_1\,\sin\varphi_2\,\cos\varphi_3 + \sin\varphi_1\,\sin\varphi_3 & \cos\varphi_1\,\sin\varphi_2\,\sin\varphi_3 + \sin\varphi_1\,\cos\varphi_3 & \cos\varphi_1\,\cos\varphi_2 \end{bmatrix}$$

(c_2) it follows from rotation of $\mathbf{e}_i$ around $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ that

$$\begin{aligned} \mathbf{R} \longrightarrow \bar{\mathbf{R}} &= \mathbf{R}_3\,\mathbf{R}_2\,\mathbf{R}_1 \\ &= \underbrace{R_{3ij}\,R_{2jo}\,R_{1oq}}_{\bar{R}_{iq}}\,(\mathbf{e}_i \otimes \mathbf{e}_q) \end{aligned}$$

with

$$\bar{R}_{iq} = \begin{bmatrix} \cos\varphi_2\,\cos\varphi_3 & \sin\varphi_1\,\sin\varphi_2\,\cos\varphi_3 - \cos\varphi_1\,\sin\varphi_3 & \cos\varphi_1\,\sin\varphi_2\,\cos\varphi_3 + \sin\varphi_1\,\sin\varphi_3 \\ \cos\varphi_2\,\sin\varphi_3 & \sin\varphi_1\,\sin\varphi_2\,\sin\varphi_3 + \cos\varphi_1\,\cos\varphi_3 & \cos\varphi_1\,\sin\varphi_2\,\sin\varphi_3 - \sin\varphi_1\,\cos\varphi_3 \\ -\sin\varphi_2 & \sin\varphi_1\,\cos\varphi_2 & \cos\varphi_1\,\cos\varphi_2 \end{bmatrix}$$

Orthogonality of "CARDANO rotation tensors":

For all $\mathbf{R} \in \{\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3, \overset{*}{\mathbf{R}}, \bar{\mathbf{R}}\}$, the following relations hold

$$\mathbf{R}^{-1} = \mathbf{R}^T, \text{ i. e. } \mathbf{R}\,\mathbf{R}^T = \mathbf{I} \quad \text{and} \quad (\det \mathbf{R})^2 = 1 \quad \longrightarrow \quad \text{orthogonality}$$

Furthermore, all rotation tensors hold the following relation

$$\det \mathbf{R} = 1 \quad : \text{ "proper" orthogonality}$$

Rem.: A basis transformation with "non-proper" orthogonal transformations ($\det \mathbf{R} = -1$) transforms a "right-handed" into a "left-handed" basis system.

Example:

here: Investigation of the orthogonality properties of $\mathbf{R}_3 = \mathbf{R}_{3ij}\,(\mathbf{e}_i \otimes \mathbf{e}_j)$

$$\text{with} \quad R_{3ij} = \begin{bmatrix} \cos\varphi_3 & -\sin\varphi_3 & 0 \\ \sin\varphi_3 & \cos\varphi_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

One looks at

$$\begin{aligned} \mathbf{R}_3\,\mathbf{R}_3^T &= R_{3ij}(\mathbf{e}_i \otimes \mathbf{e}_j)\,R_{3on}(\mathbf{e}_n \otimes \mathbf{e}_o) \\ &= R_{3ij}\,R_{3on}\,\delta_{jn}\,(\mathbf{e}_i \otimes \mathbf{e}_o) = R_{3in}\,R_{3on}\,(\mathbf{e}_i \otimes \mathbf{e}_o) \end{aligned}$$

where

$$R_{3in}\,R_{3on} = \begin{bmatrix} \sin^2\varphi_3 + \cos^2\varphi_3 & 0 & 0 \\ 0 & \sin^2\varphi_3 + \cos^2\varphi_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \delta_{io}$$

and one obtains

$$\mathbf{R}_3\,\mathbf{R}_3^T = \delta_{io}\,(\mathbf{e}_i \otimes \mathbf{e}_o) = \mathbf{I} \quad \text{q. e. d.}$$

Furthermore,

$$\det \mathbf{R}_3 := \det(R_{3ij}) = 1 \quad \longrightarrow \quad \mathbf{R}_3 \text{ is proper orthogonal}$$

Description of rotation tensors:

In general, the transformation between basis systems $\bar{\mathbf{e}}_i$ and basis systems $\overset{\circ}{\mathbf{e}}_i$ satisfies the following relation:

$$\overset{\circ}{\mathbf{e}}_i = \bar{\mathbf{R}}\,\bar{\mathbf{e}}_i \quad \text{with} \quad \bar{\mathbf{R}} = \bar{R}_{ik}\,\bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_k$$

$$\longrightarrow \quad \bar{\mathbf{e}}_i = \bar{\mathbf{R}}^T\,\overset{\circ}{\mathbf{e}}_i \quad \text{with} \quad \bar{\mathbf{R}}^{-1} \equiv \bar{\mathbf{R}}^T$$

Otherwise,

$$\bar{\mathbf{e}}_i = \overset{\circ}{\mathbf{R}}\,\overset{\circ}{\mathbf{e}}_i \quad \text{with} \quad \overset{\circ}{\mathbf{R}} = \overset{\circ}{R}_{ik}\,\overset{\circ}{\mathbf{e}}_i \otimes \overset{\circ}{\mathbf{e}}_k$$

Consequence: By comparing both relations, it follows that

$$\overset{\circ}{\mathbf{R}} = \bar{\mathbf{R}}^T, \quad \text{i. e.,} \quad \overset{\circ}{R}_{ik}\,\overset{\circ}{\mathbf{e}}_i \otimes \overset{\circ}{\mathbf{e}}_k = (\bar{R}_{ik})^T\,\bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_k \quad \longrightarrow \quad \overset{\circ}{R}_{ik} = \bar{R}_{ki}$$

In particular,

$$\begin{aligned} \overset{\circ}{\mathbf{R}} &= \overset{\circ}{R}_{ik}\,(\overset{\circ}{\mathbf{e}}_i \otimes \overset{\circ}{\mathbf{e}}_k) = \overset{\circ}{R}_{ik}\,(\bar{\mathbf{R}}\,\bar{\mathbf{e}}_i \otimes \bar{\mathbf{R}}\,\bar{\mathbf{e}}_k) \\ &= \overset{\circ}{R}_{ik}\,\bar{R}_{ni}\,\bar{\mathbf{e}}_n \otimes \bar{R}_{pk}\,\bar{\mathbf{e}}_p = (\bar{R}_{ni}\,\overset{\circ}{R}_{ik}\,\bar{R}_{pk})\,\bar{\mathbf{e}}_n \otimes \bar{\mathbf{e}}_p \overset{!}{=} \bar{R}_{pn}\,\bar{\mathbf{e}}_n \otimes \bar{\mathbf{e}}_p = \bar{\mathbf{R}}^T \end{aligned}$$

$$\longrightarrow \quad \boxed{\bar{R}_{ni}\,\overset{\circ}{R}_{ik}\,\bar{R}_{pk} \overset{!}{=} \bar{R}_{pn} \quad \longleftrightarrow \quad \bar{R}_{ni}\,\overset{\circ}{R}_{ik} = \delta_{nk}}$$

Rem.: The coefficient matrices $\bar{R}_{ni}$ and $\overset{\circ}{R}_{ik}$ are inverse to each other, i. e., in general, $\bar{R}_{ni}\,\overset{\circ}{R}_{ik} = \delta_{nk}$ implies 6 equations for the 9 unknown coefficients $\overset{\circ}{R}_{ik}$. Due to $\bar{\mathbf{R}}^{-1} = \bar{\mathbf{R}}^T$, one has $\bar{R}_{ni}^{-1} = (\bar{R}_{ni})^T = \bar{R}_{in}$, i. e.

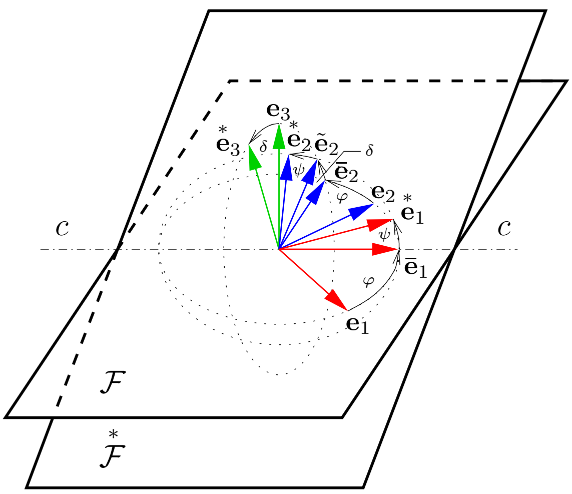
$$\boxed{\overset{\circ}{R}_{ik} = (\bar{R}_{ik})^T = \bar{R}_{ki}}$$

(C) Introduction of EULER angles

Rem.: Rotation of a basis system $\mathbf{e}_i$ around three specific axes.

Introduction of 3 specific angles around $\mathbf{e}_3$, $\bar{\mathbf{e}}_1$, $\tilde{\mathbf{e}}_3 = \overset{*}{\mathbf{e}}_3$

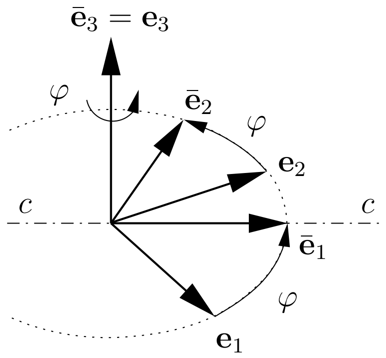
Illustration:



Idea: Given are 2 planes $\mathcal{F}$ and $\overset{*}{\mathcal{F}}$ with in-plane vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\overset{*}{\mathbf{e}}_1$, $\overset{*}{\mathbf{e}}_2$ and surface normals $\mathbf{e}_3$ and $\overset{*}{\mathbf{e}}_3$. The basis systems $\mathbf{e}_i$ and $\overset{*}{\mathbf{e}}_i$ are related to each other by the EULERian rotation tensor $\mathbf{R}$:

$$\overset{*}{\mathbf{e}}_i := \mathbf{R}\,\mathbf{e}_i$$

<u>1st step:</u>



Rotation of $\mathbf{e}_i$ in plane $\mathcal{F}$ around $\mathbf{e}_3$ with the angle φ, such that $\bar{\mathbf{e}}_i$ is directed towards $c-c$. This yields the rotation tensor

$$\mathbf{R}_3 = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{e}_j \otimes \mathbf{e}_k \, .$$

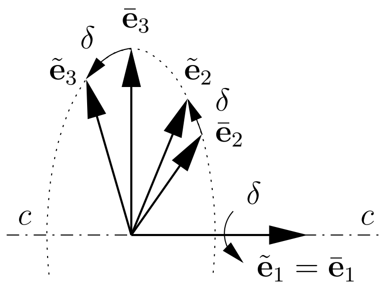
Then, the new system $\bar{\mathbf{e}}_i$ is computed as follows

$$\bar{\mathbf{e}}_i = \mathbf{R}_3\,\mathbf{e}_i = R_{3jk}\,(\mathbf{e}_j \otimes \mathbf{e}_k)\,\mathbf{e}_i = R_{3ji}\,\mathbf{e}_j \, .$$

Thus,

$$\begin{aligned} \bar{\mathbf{e}}_1 &= R_{3j1}\,\mathbf{e}_j &&= \cos\varphi\,\mathbf{e}_1 + \sin\varphi\,\mathbf{e}_2 \\ \bar{\mathbf{e}}_2 &= R_{3j2}\,\mathbf{e}_j &&= -\sin\varphi\,\mathbf{e}_1 + \cos\varphi\,\mathbf{e}_2 \\ \bar{\mathbf{e}}_3 &= R_{3j3}\,\mathbf{e}_j &&= \mathbf{e}_3 \, . \end{aligned}$$

<u>2nd step:</u>



Rotation of $\bar{\mathbf{e}}_i$ around $\bar{\mathbf{e}}_1$ with the angle δ, such that $\tilde{\mathbf{e}}_2$ lies in the plane $\overset{*}{\mathcal{F}}$, and $\tilde{\mathbf{e}}_3$ is directed normal to the plane $\overset{*}{\mathcal{F}}$. This yields the rotation tensor

$$\bar{\mathbf{R}}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\delta & -\sin\delta \\ 0 & \sin\delta & \cos\delta \end{bmatrix} \bar{\mathbf{e}}_j \otimes \bar{\mathbf{e}}_k \, .$$

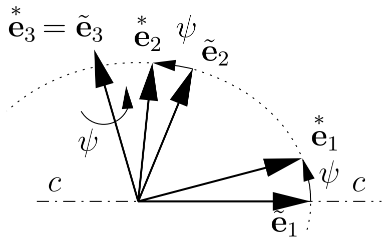
Then, the new system $\tilde{\mathbf{e}}_i$ is computed as follows

$$\tilde{\mathbf{e}}_i = \bar{\mathbf{R}}_1\,\bar{\mathbf{e}}_i = \bar{R}_{1jk}\,(\bar{\mathbf{e}}_j \otimes \bar{\mathbf{e}}_k)\,\bar{\mathbf{e}}_i = \bar{R}_{1ji}\,\bar{\mathbf{e}}_j \, .$$

Thus,

$$\begin{aligned} \tilde{\mathbf{e}}_1 &= \bar{R}_{1j1}\,\bar{\mathbf{e}}_j &&= \bar{\mathbf{e}}_1 \\ \tilde{\mathbf{e}}_2 &= \bar{R}_{1j2}\,\bar{\mathbf{e}}_j &&= \cos\delta\,\bar{\mathbf{e}}_2 + \sin\delta\,\bar{\mathbf{e}}_3 \\ \tilde{\mathbf{e}}_3 &= \bar{R}_{1j3}\,\bar{\mathbf{e}}_j &&= -\sin\delta\,\bar{\mathbf{e}}_2 + \cos\delta\,\bar{\mathbf{e}}_3 \, . \end{aligned}$$

<u>3rd step:</u>



Rotation of $\tilde{\mathbf{e}}_i$ in plane $\overset{*}{\mathcal{F}}$ around $\tilde{\mathbf{e}}_3$ with the angle ψ. This yields the rotation tensor

$$\tilde{\mathbf{R}}_3 = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tilde{\mathbf{e}}_j \otimes \tilde{\mathbf{e}}_k \, .$$

Then, the new system $\overset{*}{\mathbf{e}}_i$ is computed as follows

$$\overset{*}{\mathbf{e}}_i = \tilde{\mathbf{R}}_3\,\tilde{\mathbf{e}}_i = \tilde{R}_{3jk}\,(\tilde{\mathbf{e}}_j \otimes \tilde{\mathbf{e}}_k)\,\tilde{\mathbf{e}}_i = \tilde{R}_{3ji}\,\tilde{\mathbf{e}}_j \, .$$

Thus,

$$\begin{aligned}
\overset{*}{\mathbf{e}}_1 &= \tilde{R}_{3j1}\,\tilde{\mathbf{e}}_j = \cos\psi\,\tilde{\mathbf{e}}_1 + \sin\psi\,\tilde{\mathbf{e}}_2 \\
\overset{*}{\mathbf{e}}_2 &= \tilde{R}_{3j2}\,\tilde{\mathbf{e}}_j = -\sin\psi\,\tilde{\mathbf{e}}_1 + \cos\psi\,\tilde{\mathbf{e}}_2 \\
\overset{*}{\mathbf{e}}_3 &= \tilde{R}_{3j3}\,\tilde{\mathbf{e}}_j = \tilde{\mathbf{e}}_3\,.
\end{aligned}$$

Summary:

(a) Inserting $\tilde{\mathbf{e}}_i = \bar{\mathbf{R}}_1\,\bar{\mathbf{e}}_i$

$$\begin{aligned}
\overset{*}{\mathbf{e}}_1 &= \cos\psi\,\bar{\mathbf{e}}_1 + \sin\psi\,(\cos\delta\,\bar{\mathbf{e}}_2 + \sin\delta\,\bar{\mathbf{e}}_3) \\
\overset{*}{\mathbf{e}}_2 &= -\sin\psi\,\bar{\mathbf{e}}_1 + \cos\psi\,(\cos\delta\,\bar{\mathbf{e}}_2 + \sin\delta\,\tilde{\mathbf{e}}_3) \\
\overset{*}{\mathbf{e}}_3 &= \tilde{\mathbf{e}}_3 = -\sin\delta\,\bar{\mathbf{e}}_2 + \cos\delta\,\bar{\mathbf{e}}_3
\end{aligned}$$

Result:

$$\begin{aligned}
\overset{*}{\mathbf{e}}_1 &= \cos\psi\,\bar{\mathbf{e}}_1 + \sin\psi\,\cos\delta\,\bar{\mathbf{e}}_2 + \sin\psi\,\sin\delta\,\bar{\mathbf{e}}_3 \\
\overset{*}{\mathbf{e}}_2 &= -\sin\psi\,\bar{\mathbf{e}}_1 + \cos\psi\,\cos\delta\,\bar{\mathbf{e}}_2 + \cos\psi\,\sin\delta\,\tilde{\mathbf{e}}_3 \\
\overset{*}{\mathbf{e}}_3 &= -\sin\delta\,\bar{\mathbf{e}}_2 + \cos\delta\,\bar{\mathbf{e}}_3
\end{aligned}$$

$$\longrightarrow \quad \overset{*}{\mathbf{e}}_i = \tilde{\mathbf{R}}_3\,\underbrace{(\bar{\mathbf{R}}_1\,\bar{\mathbf{e}}_i)}_{\tilde{\mathbf{e}}_i} =: \bar{\mathbf{R}}\,\bar{\mathbf{e}}_i \qquad \text{with} \qquad \bar{\mathbf{R}} = \tilde{\mathbf{R}}_3\,\bar{\mathbf{R}}_1$$

(b) Inserting $\bar{\mathbf{e}}_i = \mathbf{R}_3\,\mathbf{e}_i$

$$\begin{aligned}
\overset{*}{\mathbf{e}}_1 &= \cos\psi\,(\cos\varphi\,\mathbf{e}_1 + \sin\varphi\,\mathbf{e}_2) + \sin\psi\,\cos\delta\,(-\sin\varphi\,\mathbf{e}_1 + \cos\varphi\,\mathbf{e}_2) + \sin\psi\,\sin\delta\,\mathbf{e}_3 \\
\overset{*}{\mathbf{e}}_2 &= -\sin\psi\,(\cos\varphi\,\mathbf{e}_1 + \sin\varphi\,\mathbf{e}_2) + \cos\psi\,\cos\delta\,(-\sin\varphi\,\mathbf{e}_1 + \cos\varphi\,\mathbf{e}_2) + \cos\psi\,\sin\delta\,\mathbf{e}_3 \\
\overset{*}{\mathbf{e}}_3 &= -\sin\delta\,(-\sin\varphi\,\mathbf{e}_1 + \cos\varphi\,\mathbf{e}_2) + \cos\delta\,\mathbf{e}_3
\end{aligned}$$

Result:

$$\begin{aligned}
\overset{*}{\mathbf{e}}_1 &= (\cos\psi\,\cos\varphi - \sin\psi\,\cos\delta\,\sin\varphi)\,\mathbf{e}_1 + \\
&\quad +(\cos\psi\,\sin\varphi + \sin\psi\,\cos\delta\,\cos\varphi)\,\mathbf{e}_2 + \sin\psi\,\sin\delta\,\mathbf{e}_3 \\
\overset{*}{\mathbf{e}}_2 &= (-\sin\psi\,\cos\varphi - \cos\psi\,\cos\delta\,\sin\varphi)\,\mathbf{e}_1 + \\
&\quad +(-\sin\psi\,\sin\varphi + \cos\psi\,\cos\delta\,\cos\varphi)\,\mathbf{e}_2 + \cos\psi\,\sin\delta\,\mathbf{e}_3 \\
\overset{*}{\mathbf{e}}_3 &= \sin\delta\,\sin\varphi\,\mathbf{e}_1 - \sin\delta\,\cos\varphi\,\mathbf{e}_2 + \cos\delta\,\mathbf{e}_3
\end{aligned}$$

$$\longrightarrow \quad \overset{*}{\mathbf{e}}_i = \bar{\mathbf{R}}\,\underbrace{(\mathbf{R}_3\,\mathbf{e}_i)}_{\bar{\mathbf{e}}_i} =: \mathbf{R}\,\mathbf{e}_i \qquad \text{with} \qquad \mathbf{R} = \bar{\mathbf{R}}\,\mathbf{R}_3 = \tilde{\mathbf{R}}_3\,\bar{\mathbf{R}}_1\,\mathbf{R}_3$$

Rotation tensors $\mathbf{R}$ and $\overset{*}{\mathbf{R}}$:

For the total rotation the following relation holds:

$$\begin{aligned}
\overset{*}{\mathbf{e}}_i &= (\tilde{\mathbf{R}}_3\,\bar{\mathbf{R}}_1\,\mathbf{R}_3)\,\mathbf{e}_i =: \mathbf{R}\,\mathbf{e}_i \\
&= (\tilde{\mathbf{R}}_3\,\bar{\mathbf{R}}_1)\,\underbrace{(\mathbf{R}_3\,\mathbf{e}_i)}_{\bar{\mathbf{e}}_i} = \tilde{\mathbf{R}}_3\,\underbrace{(\bar{\mathbf{R}}_1\,\bar{\mathbf{e}}_i)}_{\tilde{\mathbf{e}}_i} = \underbrace{\tilde{\mathbf{R}}_3\,\tilde{\mathbf{e}}_i}_{\overset{*}{\mathbf{e}}_i}
\end{aligned}$$

Furthermore,

$$\overset{*}{\mathbf{e}}_i = \mathbf{R}\,\mathbf{e}_i \quad\longrightarrow\quad \mathbf{e}_i = \mathbf{R}^T\,\overset{*}{\mathbf{e}}_i =: \overset{*}{\mathbf{R}}\,\overset{*}{\mathbf{e}}_i \quad\longrightarrow\quad \boxed{\overset{*}{\mathbf{R}} = \mathbf{R}^T}$$

Analogously to the previous considerations $\longrightarrow$ $\boxed{\overset{*}{R}_{ik} = (R_{ik})^T = R_{ki}}$

<u>Description:</u>

$$\mathbf{R} = \begin{bmatrix} \cos\psi\,\cos\varphi - \sin\psi\,\cos\delta\,\sin\varphi & -\sin\psi\,\cos\varphi - \cos\psi\,\cos\delta\,\sin\varphi & \sin\delta\,\sin\varphi \\ \cos\psi\,\sin\varphi + \sin\psi\,\cos\delta\,\cos\varphi & -\sin\psi\,\sin\varphi + \cos\psi\,\cos\delta\,\cos\varphi & -\sin\delta\,\cos\varphi \\ \sin\psi\,\sin\delta & \cos\psi\,\sin\delta & \cos\delta \end{bmatrix} \mathbf{e}_i \otimes \mathbf{e}_k$$

<u>Combining rotation tensors with different basis systems:</u>

Example: $\bar{\mathbf{R}} := \tilde{\mathbf{R}}_3\,\bar{\mathbf{R}}_1$

$$\overset{*}{\mathbf{e}}_i = \tilde{\mathbf{R}}_3\,\tilde{\mathbf{e}}_i = (\tilde{\mathbf{R}}_3\,\bar{\mathbf{R}}_1)\,\bar{\mathbf{e}}_i$$

$$\begin{aligned} \longrightarrow \quad \bar{\mathbf{R}} &= \tilde{R}_{3ik}\,(\tilde{\mathbf{e}}_i \otimes \tilde{\mathbf{e}}_k)\,\bar{R}_{1no}\,(\bar{\mathbf{e}}_n \otimes \bar{\mathbf{e}}_o) \\ &= \tilde{R}_{3ik}\,(\underbrace{\bar{\mathbf{R}}_1\,\bar{\mathbf{e}}_i \otimes \bar{\mathbf{R}}_1\,\bar{\mathbf{e}}_k}_{\bar{R}_{1si}\,\bar{\mathbf{e}}_s \otimes \bar{R}_{1tk}\,\bar{\mathbf{e}}_t})\,\bar{R}_{1no}\,(\bar{\mathbf{e}}_n \otimes \bar{\mathbf{e}}_o) \end{aligned}$$

$$\begin{aligned} \longrightarrow \quad \bar{\mathbf{R}} &= \bar{R}_{1si}\,\tilde{R}_{3ik}\,\bar{R}_{1tk}\,(\bar{\mathbf{e}}_s \otimes \bar{\mathbf{e}}_t)\,\bar{R}_{1no}\,(\bar{\mathbf{e}}_n \otimes \bar{\mathbf{e}}_o) \\ &= \bar{R}_{1si}\,\tilde{R}_{3ik}\,\bar{R}_{1tk}\,\bar{R}_{1no}\,\delta_{tn}\,(\bar{\mathbf{e}}_s \otimes \bar{\mathbf{e}}_o) \\ &= \underbrace{\bar{R}_{1si}\,\tilde{R}_{3ik}\,\bar{R}_{1tk}\,\bar{R}_{1to}}_{\bar{R}_{so}}\,(\bar{\mathbf{e}}_s \otimes \bar{\mathbf{e}}_o) \end{aligned}$$

Thus, the rotation tensor $\bar{\mathbf{R}}$ is given by

$$\bar{\mathbf{R}} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi\,\cos\delta & \cos\psi\,\cos\delta & -\sin\delta \\ \sin\psi\,\sin\delta & \cos\psi\,\sin\delta & \cos\delta \end{bmatrix} \bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_k$$

Rem.: Concerning CARDANO angles, all partial rotations (e. g. $\mathbf{R} = \mathbf{R}_3\,\mathbf{R}_2\,\mathbf{R}_1$ with $\overset{*}{\mathbf{e}}_i = \mathbf{R}\,\mathbf{e}_i$) are carried out with respect to the same basis $\mathbf{e}_i$, i. e. the combination of the partial rotations is much easier.

Rotation around a fixed axis:

Rem.: A rotation around 3 independent axes can also be described by a rotation around the resulting axis of rotation:

$\longrightarrow$ EULER-RODRIGUES representation of the spatial rotation

The EULER-RODRIGUES representation of the rotation is discussed later (see section 2.7).

2.5 Higher order tensors

Definition: An arbitrary n-th order tensor is given by

$$\overset{n}{\mathbf{A}} \in \mathcal{V}^3 \otimes \mathcal{V}^3 \otimes \cdots \otimes \mathcal{V}^3 \quad (n \text{ times})$$
$$\text{with } \mathcal{V}^3 \otimes \mathcal{V}^3 \otimes \cdots \otimes \mathcal{V}^3 \quad : n\text{-th order dyadic product space}$$

Rem.: Usually, $n \geq 2$. However, there exist special cases for $n = 1$ (vector) and $n = 0$ (scalar).

General description of the linear mapping

Definition: A linear mapping is a "contracting product" (contraction) given by

$$\overset{n}{\mathbf{A}}\,\overset{s}{\mathbf{B}} = \overset{n-s}{\mathbf{C}} \quad \text{with} \quad n \geq s$$

Descriptive example on simple tensors:

$$\underbrace{(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} \otimes \mathbf{d})}_{\overset{4}{\mathbf{A}}}\,\underbrace{(\mathbf{e} \otimes \mathbf{f})}_{\mathbf{B}} = \underbrace{(\mathbf{c} \cdot \mathbf{e})\,(\mathbf{d} \cdot \mathbf{f})\,\mathbf{a} \otimes \mathbf{b}}_{\mathbf{C}}$$

Fundamental 4-th order tensors

Rem.: 4-th order fundamental tensors are built by a dyadic product of 2nd order identity tensors and the corresponding independent transpositions.

One introduces:

$$\begin{aligned}
\mathbf{I} \otimes \mathbf{I} &= (\mathbf{e}_i \otimes \mathbf{e}_i) \otimes (\mathbf{e}_j \otimes \mathbf{e}_j) \\
(\mathbf{I} \otimes \mathbf{I})^{\overset{23}{T}} &= \mathbf{e}_i \otimes \mathbf{e}_j \,\otimes\, \mathbf{e}_i \otimes \mathbf{e}_j \\
(\mathbf{I} \otimes \mathbf{I})^{\overset{24}{T}} &= \mathbf{e}_i \otimes \mathbf{e}_j \,\otimes\, \mathbf{e}_j \otimes \mathbf{e}_i
\end{aligned}$$

with $(\,\cdot\,)^{\overset{ik}{T}}$: transposition, defined by the exchange of the i-th and the k-th basis system

Rem.: Further transpositions of $\mathbf{I} \otimes \mathbf{I}$ do not lead to further independent tensors. The fundamental tensors from above exhibit the property

$$\overset{4}{\mathbf{A}} = \overset{4}{\mathbf{A}}{}^T \quad \text{with} \quad \overset{4}{\mathbf{A}}{}^T = (\overset{4}{\mathbf{A}}{}^{\overset{13}{T}})^{\overset{24}{T}}$$

Consequence: The 4-th order fundamental tensors are symmetric (concerning an exchange of the first two and the second two basis systems).

Properties of 4-th order fundamental tensors

(a) identical map

$$\begin{aligned}(\mathbf{I}\otimes\mathbf{I})^{\overset{23}{T}}\,\mathbf{A} &= (\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_i\otimes\mathbf{e}_j)\,a_{st}(\mathbf{e}_s\otimes\mathbf{e}_t)\\ &= a_{st}\,\delta_{is}\,\delta_{jt}\,(\mathbf{e}_i\otimes\mathbf{e}_j) = a_{ij}\,(\mathbf{e}_i\otimes\mathbf{e}_j) = \mathbf{A}\end{aligned}$$

$$\longrightarrow \quad \overset{4}{\mathbf{I}} := (\mathbf{I}\otimes\mathbf{I})^{\overset{23}{T}} \quad \text{is 4-th order identity tensor}$$

(b) “transposing” map

$$\begin{aligned}(\mathbf{I}\otimes\mathbf{I})^{\overset{24}{T}}\,\mathbf{A} &= (\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_j\otimes\mathbf{e}_i)\,a_{st}\,(\mathbf{e}_s\otimes\mathbf{e}_t)\\ &= a_{st}\,\delta_{js}\,\delta_{it}\,(\mathbf{e}_i\otimes\mathbf{e}_j) = a_{ji}\,(\mathbf{e}_i\otimes\mathbf{e}_j) = \mathbf{A}^T\end{aligned}$$

(c) “tracing” map

$$\begin{aligned}(\mathbf{I}\otimes\mathbf{I})\,\mathbf{A} &= (\mathbf{e}_i\otimes\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_j)\,a_{st}\,(\mathbf{e}_s\otimes\mathbf{e}_t)\\ &= a_{st}\,\delta_{js}\,\delta_{jt}\,(\mathbf{e}_i\otimes\mathbf{e}_i) = a_{jj}\,(\mathbf{e}_i\otimes\mathbf{e}_i)\\ &= (\mathbf{A}\cdot\mathbf{I})\,\mathbf{I} = (\operatorname{tr}\mathbf{A})\,\mathbf{I}\end{aligned}$$

$$\text{with} \quad \mathbf{A}\cdot\mathbf{I} = a_{st}\,(\mathbf{e}_s\otimes\mathbf{e}_t)\cdot(\mathbf{e}_j\otimes\mathbf{e}_j) = a_{st}\,\delta_{sj}\,\delta_{tj} = a_{jj}$$

Specific 4-th order tensors

Let $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$ be arbitrary 2nd order tensors. Then, a 4-th order tensor $\overset{4}{\mathbf{A}}$ can be defined exhibiting the following properties:

$$\begin{aligned}\overset{4}{\mathbf{A}} &= (\mathbf{A}\otimes\mathbf{B})^{\overset{23}{T}} = (\mathbf{B}^T\otimes\mathbf{A}^T)^{\overset{14}{T}} \qquad (*)\\ \overset{4}{\mathbf{A}}{}^T &= [(\mathbf{A}\otimes\mathbf{B})^{\overset{23}{T}}]^T = (\mathbf{A}^T\otimes\mathbf{B}^T)^{\overset{23}{T}}\\ \overset{4}{\mathbf{A}}{}^{-1} &= [(\mathbf{A}\otimes\mathbf{B})^{\overset{23}{T}}]^{-1} = (\mathbf{A}^{-1}\otimes\mathbf{B}^{-1})^{\overset{23}{T}}\end{aligned}$$

Furthermore, following relation holds:

$$(\overset{4}{\cdot})^T = [(\overset{4}{\cdot})^{\overset{13}{T}}]^{\overset{24}{T}}$$

From $(*)$, the following relations can be derived:

$$\begin{aligned}(\mathbf{A}\otimes\mathbf{B})^{\overset{23}{T}}(\mathbf{C}\otimes\mathbf{D})^{\overset{23}{T}} &= (\mathbf{AC}\otimes\mathbf{BD})^{\overset{23}{T}}\\ (\mathbf{A}\otimes\mathbf{B})^{\overset{23}{T}}(\mathbf{C}\otimes\mathbf{D}) &= (\mathbf{ACB}^T\otimes\mathbf{D})\\ (\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D})^{\overset{23}{T}} &= (\mathbf{A}\otimes\mathbf{C}^T\mathbf{BD})\end{aligned}$$

and

$$\begin{aligned}(\mathbf{A}\otimes\mathbf{B})^{\overset{23}{T}}\mathbf{C} &= \mathbf{ACB}^T\\ (\mathbf{A}\otimes\mathbf{B})^{\overset{23}{T}}\mathbf{v} &= [\mathbf{A}\otimes(\mathbf{B}\,\mathbf{v})]^{\overset{23}{T}}\end{aligned}$$

Defining a 4-th order tensor $\overset{4}{\mathbf{B}}$ with the properties

$$
\begin{aligned}
\overset{4}{\mathbf{B}} &= (\mathbf{A} \otimes \mathbf{B})^{\overset{24}{T}} = [(\mathbf{A} \otimes \mathbf{B})^{\overset{13}{T}}]^T \\
\overset{4}{\mathbf{B}}{}^T &= [(\mathbf{A} \otimes \mathbf{B})^{\overset{24}{T}}]^T = (\mathbf{B} \otimes \mathbf{A})^{\overset{24}{T}} \\
\overset{4}{\mathbf{B}}{}^{-1} &= [(\mathbf{A} \otimes \mathbf{B})^{\overset{24}{T}}]^{-1} = (\mathbf{B}^{T-1} \otimes \mathbf{A}^{T-1})^{\overset{24}{T}}
\end{aligned}
$$

it can be shown that

$$
\begin{aligned}
(\mathbf{A} \otimes \mathbf{B})^{\overset{24}{T}}(\mathbf{C} \otimes \mathbf{D})^{\overset{24}{T}} &= (\mathbf{A}\mathbf{D}^T \otimes \mathbf{B}^T\mathbf{C})^{\overset{23}{T}} \\
(\mathbf{A} \otimes \mathbf{B})^{\overset{23}{T}}(\mathbf{C} \otimes \mathbf{D})^{\overset{24}{T}} &= (\mathbf{A}\mathbf{C} \otimes \mathbf{D}\mathbf{B}^T)^{\overset{24}{T}} \\
(\mathbf{A} \otimes \mathbf{B})^{\overset{24}{T}}(\mathbf{C} \otimes \mathbf{D})^{\overset{23}{T}} &= (\mathbf{A}\mathbf{D} \otimes \mathbf{C}^T\mathbf{B})^{\overset{24}{T}} \\
(\mathbf{A} \otimes \mathbf{B})^{\overset{24}{T}}(\mathbf{C} \otimes \mathbf{D}) &= (\mathbf{A}\mathbf{C}^T\mathbf{B} \otimes \mathbf{D}) \\
(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D})^{\overset{24}{T}} &= (\mathbf{A} \otimes \mathbf{D}\mathbf{B}^T\mathbf{C})
\end{aligned}
$$

and

$$
(\mathbf{A} \otimes \mathbf{B})^{\overset{24}{T}}\mathbf{C} = \mathbf{A}\mathbf{C}^T\mathbf{B}
$$

Furthermore, the following relation holds:

$$
(\overset{4}{\mathbf{C}}\overset{4}{\mathbf{D}})^T = \overset{4}{\mathbf{D}}{}^T\overset{4}{\mathbf{C}}{}^T
$$

where $\overset{4}{\mathbf{C}}$ and $\overset{4}{\mathbf{D}}$ are arbitrary 4-th order tensors.

High order tensors and incomplete mappings

If higher order tensors are applied to other tensors in the sense of incomplete mappings, one has to know how many of the basis vectors have to be linked by scalar products. Therefore, a underlined supercript $(\cdot)^{\underline{i}}$ indicates the order of the desired result after the tensor operation has been carried out.

Examples in basis notation:

$$
\begin{aligned}
(\overset{4}{\mathbf{A}}\,\overset{3}{\mathbf{B}})^{\underline{3}} &= [a_{ijkl}\,(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l)\,b_{mno}\,(\mathbf{e}_m \otimes \mathbf{e}_n \otimes \mathbf{e}_o)]^{\underline{3}} \\
&= a_{ijkl}\,b_{mno}\,\delta_{km}\,\delta_{ln}\,(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_o) \\
(\mathbf{A}\,\overset{3}{\mathbf{B}})^{\underline{1}} &= [a_{ij}\,(\mathbf{e}_i \otimes \mathbf{e}_j)\,b_{mno}\,(\mathbf{e}_m \otimes \mathbf{e}_n \otimes \mathbf{e}_o)]^{\underline{1}} \\
&= a_{ij}\,b_{mno}\,\delta_{im}\,\delta_{jn}\,\mathbf{e}_o
\end{aligned}
$$

Note:	Note in passing that the incomplete mapping is governed by scalar products of a sufficient number of inner basis systems.

2.6 Fundamental tensor of 3rd order (Ricci permutation tensor)

Rem.: The fundamental tensor of 3rd order is introduced in the context of the "outer product" (e. g. vector product between vectors).

Definition: The fundamental tensor $\overset{3}{\mathbf{E}}$ satisfies the rule

$$\mathbf{u} \times \mathbf{v} = \overset{3}{\mathbf{E}}\ (\mathbf{u} \otimes \mathbf{v})$$

Introduction of $\overset{3}{\mathbf{E}}$ in basis notation:

There is

$$\overset{3}{\mathbf{E}} = e_{ijk}\,(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k)$$

with the "permutation symbol" e_{ijk}

$$e_{ijk} = \begin{cases} 1 & : \text{even permutation} \\ -1 & : \text{odd permutation} \\ 0 & : \text{double indexing} \end{cases} \longrightarrow \begin{cases} e_{123} = e_{231} = e_{312} = 1 \\ e_{321} = e_{213} = e_{132} = -1 \\ \text{all remaining } e_{ijk} \text{ vanish} \end{cases}$$

Application of $\overset{3}{\mathbf{E}}$ to the vector product of vectors:

From the above definition,

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \overset{3}{\mathbf{E}}\ (\mathbf{u} \otimes \mathbf{v}) \\ &= e_{ijk}\,(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k)\,(u_s\,\mathbf{e}_s \otimes v_t\,\mathbf{e}_t) \\ &= e_{ijk}\,u_s\,v_t\,\delta_{js}\,\delta_{kt}\,\mathbf{e}_i = e_{ijk}\,u_j\,v_k\,\mathbf{e}_i \\ &= (u_2\,v_3 - u_3\,v_2)\,\mathbf{e}_1 + (u_3\,v_1 - u_1\,v_3)\,\mathbf{e}_2 + (u_1\,v_2 - u_2\,v_1)\,\mathbf{e}_3 \end{aligned}$$

Comparison with the computation by use of the matrix notation, cf. page 5

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \cdots \quad \text{q. e. d.}$$

An identity for $\overset{3}{\mathbf{E}}$:

Incomplete mapping of two Ricci-tensors yielding a 2nd or 4th order object

$$(\overset{3}{\mathbf{E}}\,\overset{3}{\mathbf{E}})^{\underline{2}} = 2\,\mathbf{I}\,, \quad (\overset{3}{\mathbf{E}}\,\overset{3}{\mathbf{E}})^{\underline{4}} = (\,\mathbf{I} \otimes \mathbf{I}\,)^{\overset{23}{T}} - (\,\mathbf{I} \otimes \mathbf{I}\,)^{\overset{24}{T}}$$

2.7 The axial vector

Rem.: The axial vector (pseudo vector) can be used for the description of rotations (rotation vector).

Definition: The axial vector $\overset{A}{\mathbf{t}}$ is associated with the skew-symmetric part $\operatorname{skw}\mathbf{T}$ of an arbitrary tensor $\mathbf{T} \in \mathcal{V}^3 \otimes \mathcal{V}^3$ via

$$\overset{A}{\mathbf{t}} := \tfrac{1}{2} \overset{3}{\mathbf{E}} \; \mathbf{T}^T$$

One calculates,

$$\begin{aligned} \overset{A}{\mathbf{t}} &= \tfrac{1}{2}\, e_{ijk}\, (\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k)\, t_{st}\, (\mathbf{e}_t \otimes \mathbf{e}_s) \\ &= \tfrac{1}{2}\, e_{ijk}\, t_{st}\, \delta_{jt}\, \delta_{ks}\, \mathbf{e}_i = \tfrac{1}{2}\, e_{ijk}\, t_{kj}\, \mathbf{e}_i \\ &= \tfrac{1}{2}\left[(t_{32} - t_{23})\, \mathbf{e}_1 + (t_{13} - t_{31})\, \mathbf{e}_2 + (t_{21} - t_{12})\, \mathbf{e}_3\right] \end{aligned}$$

It follows from 2.3 (b)

$$\mathbf{T} = \operatorname{sym}\mathbf{T} + \operatorname{skw}\mathbf{T}$$

Thus, the axial vector of $\mathbf{T}$ is given by

$$\begin{aligned} \overset{A}{\mathbf{t}} &= \tfrac{1}{2} \overset{3}{\mathbf{E}} \, (\operatorname{sym}\mathbf{T} + \operatorname{skw}\mathbf{T})^T \\ &= \tfrac{1}{2} \overset{3}{\mathbf{E}} \, (\operatorname{skw}\mathbf{T}^T) = -\tfrac{1}{2} \overset{3}{\mathbf{E}} \, (\operatorname{skw}\mathbf{T}) \end{aligned}$$

Rem.: A symmetric tensor has no axial vector.

Axial vector and linear mapping:

The following relation holds:

$$(\operatorname{skw}\mathbf{T})\, \mathbf{v} = \overset{A}{\mathbf{t}} \times \mathbf{v} \quad \forall\, \mathbf{v} \in \mathcal{V}^3$$

Axial vector and the vector product of tensors:

Definition: The vector product of 2 tensors $\{\mathbf{T}, \mathbf{S}\} \in \mathcal{V}^3 \otimes \mathcal{V}^3$ satisfies

$$\mathbf{S} \times \mathbf{T} = \overset{3}{\mathbf{E}} \; (\mathbf{S}\, \mathbf{T}^T)$$

Rem.: The vector product (cross product) of 2 tensors yields a vector.

In comparison with the definition of the axial vector follows

$$\mathbf{I} \times \mathbf{T} = \overset{3}{\mathbf{E}} \; \mathbf{T}^T = 2 \; \overset{A}{\mathbf{t}}$$

Furthermore, the vector product of 2 tensors yields

$$\mathbf{S} \times \mathbf{T} = -\mathbf{T} \times \mathbf{S}$$

Axial vector and outer tensor product of vector and tensor:

Definition: The outer tensor product of a vector $\mathbf{u} \in \mathcal{V}^3$ and a tensor $\mathbf{T} \in \mathcal{V}^3 \otimes \mathcal{V}^3$ satisfies

$$(\mathbf{u} \times \mathbf{T})\,\mathbf{v} = \mathbf{u} \times (\mathbf{T}\,\mathbf{v})\;; \quad \mathbf{v} \in \mathcal{V}^3$$

Rem.: The outer tensor product of vector and tensor yields a tensor.

The following relations hold:

$$\mathbf{u} \times \mathbf{T} = -(\mathbf{u} \times \mathbf{T})^T = -\mathbf{T} \times \mathbf{u}$$

$\longrightarrow$ i. e. $\mathbf{u} \times \mathbf{T}$ is skew-symmetric

$$\mathbf{u} \times \mathbf{T} = [\overset{3}{\mathbf{E}}\ (\mathbf{u} \otimes \mathbf{T})]^{\underline{2}}$$

with $(\,\cdot\,)^{\underline{2}}$: "incomplete" linear mapping (association) resulting in a 2nd order tensor.

Evaluation in basis notation leads to

$$\begin{aligned}\mathbf{u} \times \mathbf{T} &= [(e_{ijk}\,\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k)\,(u_r\,\mathbf{e}_r \otimes t_{st}\,\mathbf{e}_s \otimes \mathbf{e}_t)]^{\underline{2}}\\ &= e_{ijk}\,u_r\,t_{st}\,\delta_{jr}\,\delta_{ks}\,(\mathbf{e}_i \otimes \mathbf{e}_t)\\ &= e_{ijk}\,u_j\,t_{kt}\,(\mathbf{e}_i \otimes \mathbf{e}_t)\end{aligned}$$

In particular, if $\mathbf{T} \equiv \mathbf{I}$, the following relation holds:

$$\mathbf{u} \times \mathbf{I} = [\overset{3}{\mathbf{E}}\ (\mathbf{u} \otimes \mathbf{I})]^{\underline{2}} = e_{ijk}\,u_j\,\delta_{kt}\,(\mathbf{e}_i \otimes \mathbf{e}_t) = e_{ijt}\,u_j\,(\mathbf{e}_i \otimes \mathbf{e}_t)$$

Furthermore, for the special tensor $\mathbf{u} \times \mathbf{I}$ follows

$$\overset{3}{\mathbf{E}}\ (\mathbf{u} \times \mathbf{I}) = -2\,\mathbf{u}$$

$$\longrightarrow \quad \mathbf{u} = -\tfrac{1}{2}\,\overset{3}{\mathbf{E}}\ (\mathbf{u} \times \mathbf{I}) = \tfrac{1}{2}\,\overset{3}{\mathbf{E}}\ (\mathbf{u} \times \mathbf{I})^T$$

Consequence: In the tensor $\mathbf{u} \times \mathbf{I}$, the vector $\mathbf{u}$ is already the corresponding axial vector.

Finally, the following relation holds:

$$\mathbf{u} \times \mathbf{I} = -\,\overset{3}{\mathbf{E}}\ \mathbf{u}$$

$$\longrightarrow \quad \overset{3}{\mathbf{E}}\ (\mathbf{u} \times \mathbf{I}) = -\,\overset{3}{\mathbf{E}}\ (\overset{3}{\mathbf{E}}\ \mathbf{u}) = -(\overset{3}{\mathbf{E}}\,\overset{3}{\mathbf{E}})^{\underline{2}}\,\mathbf{u} \overset{!}{=} -2\,\mathbf{u}$$

$$\text{i. e.} \quad (\overset{3}{\mathbf{E}}\,\overset{3}{\mathbf{E}})^{\underline{2}} = 2\,\mathbf{I}$$

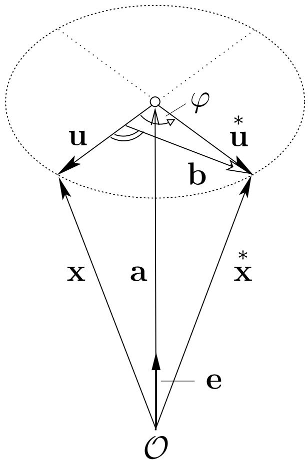
Some additional rules:

$$(\mathbf{a} \times \mathbf{b}) \otimes \mathbf{c} = \mathbf{a} \times (\mathbf{b} \otimes \mathbf{c})$$

$$(\mathbf{I} \times \mathbf{T}) \cdot \mathbf{w} = \mathbf{T} \cdot \boldsymbol{\Omega} \quad \text{with} \quad \boldsymbol{\Omega} = \mathbf{w} \times \mathbf{I}$$

Application to the tensor product of vector and tensor

Rotation around a fixed spatial axis



Rotation of $\mathbf{x}$ around axis $\mathbf{e}$

$$\overset{*}{\mathbf{x}} = \mathbf{a} + \overset{*}{\mathbf{u}} = \mathbf{a} + C_1\,\mathbf{u} + \mathbf{b}$$

$$\text{with} \quad \begin{cases} \mathbf{a} = (\mathbf{x} \cdot \mathbf{e})\,\mathbf{e} \\ \mathbf{u} = \mathbf{x} - \mathbf{a} \\ \mathbf{b} = C_2\,(\mathbf{e} \times \mathbf{x}) \end{cases}$$

$$\text{and} \quad \boldsymbol{\varphi} = \varphi\,\mathbf{e}\,; \quad |\mathbf{e}| = 1$$

Determination of the constants C_1 and C_2:

(a) For the angle between $\mathbf{u}$ and $\overset{*}{\mathbf{u}}$, the following relation holds

$$\cos\varphi = \frac{\mathbf{u} \cdot \overset{*}{\mathbf{u}}}{|\mathbf{u}|\,|\overset{*}{\mathbf{u}}|} \qquad \text{with} \quad |\mathbf{u}| = |\overset{*}{\mathbf{u}}|$$

Furthermore, the following relation holds

$$\mathbf{u} \cdot \overset{*}{\mathbf{u}} = \mathbf{u} \cdot (C_1\,\mathbf{u} + \mathbf{b}) = C_1\,\mathbf{u} \cdot \mathbf{u} + \underbrace{\mathbf{u} \cdot \mathbf{b}}_{= 0,\ \text{da}\ \mathbf{u} \perp \mathbf{b}} = C_1\,|\mathbf{u}|^2$$

Thus,

$$\cos\varphi = \frac{C_1\,|\mathbf{u}|^2}{|\mathbf{u}|^2} = C_1 \quad \longrightarrow \quad C_1 = \cos\varphi$$

(b) For the angle between $\mathbf{b}$ and $\overset{*}{\mathbf{u}}$, the following relation holds

$$\cos(90° - \varphi) = \sin\varphi = \frac{\mathbf{b} \cdot \overset{*}{\mathbf{u}}}{|\mathbf{b}|\,|\overset{*}{\mathbf{u}}|}$$

Furthermore, the following relation holds

$$\mathbf{b} \cdot \overset{*}{\mathbf{u}} = \mathbf{b} \cdot (C_1\,\mathbf{u} + \mathbf{b}) = C_1\,\underbrace{\mathbf{b} \cdot \mathbf{u}}_{= 0,\ \text{da}\ \mathbf{u} \perp \mathbf{b}} + \mathbf{b} \cdot \mathbf{b} = |\mathbf{b}|^2$$

and

$$|\mathbf{b}| = C_2\,|\mathbf{e} \times \mathbf{x}| = C_2\,\underbrace{|\mathbf{e}|}_{1}\,\underbrace{|\mathbf{x}|\sin\sphericalangle(\mathbf{e}\,;\,\mathbf{x})}_{|\mathbf{u}|} = C_2\,|\mathbf{u}|$$

Thus, leading to

$$\sin\varphi = \frac{|\mathbf{b}|^2}{|\mathbf{b}|\,|\mathbf{u}|} = \frac{|\mathbf{b}|}{|\mathbf{u}|} = \frac{C_2\,|\mathbf{u}|}{|\mathbf{u}|} = C_2 \quad \longrightarrow \quad C_2 = \sin\varphi$$

Thus, $\overset{*}{\mathbf{x}}$ is given by

$$\overset{*}{\mathbf{x}} = (\mathbf{x}\cdot\mathbf{e})\,\mathbf{e} + \cos\varphi\,[\mathbf{x} - (\mathbf{x}\cdot\mathbf{e})\,\mathbf{e}] + \sin\varphi\,(\mathbf{e}\times\mathbf{x})$$

Determination of the rotation tensor R:

For the tensor product of vector and tensor, the following relation holds:

$$(\mathbf{e}\times\mathbf{I})\,\mathbf{x} = \mathbf{e}\times(\mathbf{I}\,\mathbf{x}) = \mathbf{e}\times\mathbf{x}$$

Thus,

$$\overset{*}{\mathbf{x}} = (\mathbf{e}\otimes\mathbf{e})\,\mathbf{x} + \cos\varphi\,(\mathbf{I}-\mathbf{e}\otimes\mathbf{e})\,\mathbf{x} + \sin\varphi\,(\mathbf{e}\times\mathbf{I})\,\mathbf{x} \overset{!}{=} \mathbf{R}\,\mathbf{x}$$

$$\longrightarrow \quad \boxed{\mathbf{R} = \mathbf{e}\otimes\mathbf{e} + \cos\varphi\,(\mathbf{I}-\mathbf{e}\otimes\mathbf{e}) + \sin\varphi\,(\mathbf{e}\times\mathbf{I})} \qquad (*)$$

Rem.: $(*)$ is the EULER-RODRIGUES form of the spatial rotation.

Example: Rotation with φ_3 around the $\mathbf{e}_3$ axis

$$\mathbf{R} = \mathbf{R}_3 = \mathbf{e}_3\otimes\mathbf{e}_3 + \cos\,\varphi_3\,(\mathbf{I}-\mathbf{e}_3\otimes\mathbf{e}_3) + \sin\,\varphi_3\,(\mathbf{e}_3\times\mathbf{I})$$

The following relation holds:

$$\begin{aligned}
\mathbf{e}_3\times\mathbf{I} &= [\overset{3}{\mathbf{E}}\,(\mathbf{e}_3\otimes\mathbf{I})]^{\underline{2}} \\
&= [e_{ijk}\,(\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k)\,(\mathbf{e}_3\otimes\mathbf{e}_l\otimes\mathbf{e}_l)]^{\underline{2}} \\
&= e_{ijk}\,\delta_{j3}\,\delta_{kl}\,(\mathbf{e}_i\otimes\mathbf{e}_l) = e_{i3l}\,(\mathbf{e}_i\otimes\mathbf{e}_l) \\
&= \mathbf{e}_2\otimes\mathbf{e}_1 - \mathbf{e}_1\otimes\mathbf{e}_2
\end{aligned}$$

Thus, leading to

$$\begin{aligned}
\mathbf{R}_3 \quad &= \quad \mathbf{e}_3\otimes\mathbf{e}_3 + \cos\varphi_3\,(\mathbf{e}_1\otimes\mathbf{e}_1 + \mathbf{e}_2\otimes\mathbf{e}_2) + \sin\varphi_3\,(\mathbf{e}_2\otimes\mathbf{e}_1 - \mathbf{e}_1\otimes\mathbf{e}_2) \\
&= \quad R_{3ij}\,(\mathbf{e}_i\otimes\mathbf{e}_j)
\end{aligned}$$

$$\text{with} \qquad R_{3ij} = \begin{bmatrix} \cos\varphi_3 & -\sin\varphi_3 & 0 \\ \sin\varphi_3 & \cos\varphi_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{q. e. d.}$$

2.8 The outer tensor product of tensors

Definition: The outer tensor product of tensors (double cross product) is defined via

$$(\mathbf{A} \mathbin{⨳} \mathbf{B})(\mathbf{u}_1 \times \mathbf{u}_2) := \mathbf{A}\mathbf{u}_1 \times \mathbf{B}\mathbf{u}_2 - \mathbf{A}\mathbf{u}_2 \times \mathbf{B}\mathbf{u}_1$$

As a direct consequence, one finds

$$\mathbf{A} \mathbin{⨳} \mathbf{B} = \mathbf{B} \mathbin{⨳} \mathbf{A}$$

Furthermore, the following relations hold:

$$\begin{aligned}
(\mathbf{A} \mathbin{⨳} \mathbf{B})^T &= \mathbf{A}^T \mathbin{⨳} \mathbf{B}^T \\
(\mathbf{A} \mathbin{⨳} \mathbf{B})\,(\mathbf{C} \mathbin{⨳} \mathbf{D}) &= (\mathbf{A}\,\mathbf{C} \mathbin{⨳} \mathbf{B}\,\mathbf{D}) + (\mathbf{A}\,\mathbf{D} \mathbin{⨳} \mathbf{B}\,\mathbf{C}) \\
(\mathbf{I} \mathbin{⨳} \mathbf{I}) &= 2\,\mathbf{I} \\
(\mathbf{a} \otimes \mathbf{b}) \mathbin{⨳} (\mathbf{c} \otimes \mathbf{d}) &= (\mathbf{a} \times \mathbf{c}) \otimes (\mathbf{b} \times \mathbf{d}) \\
(\mathbf{A} \mathbin{⨳} \mathbf{B}) \cdot \mathbf{C} &= (\mathbf{B} \mathbin{⨳} \mathbf{C}) \cdot \mathbf{A} = (\mathbf{C} \mathbin{⨳} \mathbf{A}) \cdot \mathbf{B}
\end{aligned}$$

From the above definition, it is easily proved that

$$[(\mathbf{A} \mathbin{⨳} \mathbf{B}) \cdot \mathbf{C}][(\mathbf{u}_1 \times \mathbf{u}_2) \cdot \mathbf{u}_3] = e_{ijk}\,(\mathbf{A}\mathbf{u}_i \times \mathbf{B}\mathbf{u}_j) \cdot \mathbf{C}\mathbf{u}_k$$

The outer tensor product in basis notation

$$\begin{aligned}
\mathbf{A} \mathbin{⨳} \mathbf{B} &= a_{ik}\,(\mathbf{e}_i \otimes \mathbf{e}_k) \mathbin{⨳} b_{no}\,(\mathbf{e}_n \otimes \mathbf{e}_o) \\
&= a_{ik}\,b_{no}\,(\mathbf{e}_i \times \mathbf{e}_n) \otimes (\mathbf{e}_k \times \mathbf{e}_o)
\end{aligned}$$

$$\text{with} \quad \begin{cases} \mathbf{e}_i \times \mathbf{e}_n = \overset{3}{\mathbf{E}}\,(\mathbf{e}_i \otimes \mathbf{e}_n) = e_{inj}\,\mathbf{e}_j \\ \mathbf{e}_k \times \mathbf{e}_o = \overset{3}{\mathbf{E}}\,(\mathbf{e}_k \otimes \mathbf{e}_o) = e_{kop}\,\mathbf{e}_p \end{cases}$$

$$\longrightarrow \quad \mathbf{A} \mathbin{⨳} \mathbf{B} = a_{ik}\,b_{no}\,e_{inj}\,e_{kop}\,(\mathbf{e}_j \otimes \mathbf{e}_p)$$

Furthermore, it follows that

$$\begin{aligned}
\mathbf{A} \mathbin{⨳} \mathbf{I} &= (\mathbf{A} \cdot \mathbf{I})\,\mathbf{I} - \mathbf{A}^T \\
\mathbf{A} \mathbin{⨳} \mathbf{B} &= (\mathbf{A} \cdot \mathbf{I})\,(\mathbf{B} \cdot \mathbf{I})\,\mathbf{I} - (\mathbf{A}^T \cdot \mathbf{B})\,\mathbf{I} - (\mathbf{A} \cdot \mathbf{I})\,\mathbf{B}^T - \\
&\quad -(\mathbf{B} \cdot \mathbf{I})\,\mathbf{A}^T + \mathbf{A}^T\,\mathbf{B}^T + \mathbf{B}^T\,\mathbf{A}^T \\
(\mathbf{A} \mathbin{⨳} \mathbf{B}) \cdot \mathbf{C} &= (\mathbf{A} \cdot \mathbf{I})\,(\mathbf{B} \cdot \mathbf{I})\,(\mathbf{C} \cdot \mathbf{I}) - (\mathbf{A} \cdot \mathbf{I})\,(\mathbf{B}^T \cdot \mathbf{C}) - (\mathbf{B} \cdot \mathbf{I})\,(\mathbf{A}^T \cdot \mathbf{C}) - \\
&\quad -(\mathbf{C} \cdot \mathbf{I})\,(\mathbf{A}^T \cdot \mathbf{B}) + (\mathbf{A}^T\,\mathbf{B}^T) \cdot \mathbf{C} + (\mathbf{B}^T\,\mathbf{A}^T) \cdot \mathbf{C}
\end{aligned}$$

The cofactor, the adjoint tensor and the determinant:

The following relations hold:

$$\operatorname{cof}\mathbf{A} = \tfrac{1}{2}\,\mathbf{A} \,\#\, \mathbf{A} =: \overset{+}{\mathbf{A}}, \quad \operatorname{adj}\mathbf{A} = (\operatorname{cof}\mathbf{A})^T$$

$$\det\mathbf{A} = \tfrac{1}{6}\,(\mathbf{A} \,\#\, \mathbf{A})\cdot\mathbf{A} = \det|a_{ik}| = \frac{(\mathbf{A}\mathbf{u}_1 \times \mathbf{A}\mathbf{u}_2)\cdot\mathbf{A}\mathbf{u}_3}{(\mathbf{u}_1 \times \mathbf{u}_2)\cdot\mathbf{u}_3}$$

In basis notation the following relation holds:

$$\overset{+}{\mathbf{A}} = \tfrac{1}{2}\,(a_{ik}\,a_{no}\,e_{inj}\,e_{kop})\,(\mathbf{e}_j \otimes \mathbf{e}_p) = \overset{+}{a}_{jp}\;(\mathbf{e}_j \otimes \mathbf{e}_p)$$

Rem.: The coefficient matrix $\overset{+}{a}_{jp}$ of the cofactor $\operatorname{cof}\mathbf{A}$ contains at each position $(\,\cdot\,)_{jp}$ the corresponding subdeterminant of $\mathbf{A}$

$$\overset{+}{a}_{11} = a_{22}\,a_{33} - a_{23}\,a_{32} \quad \text{etc.}$$

The inverse tensor:

The following relation holds:

$$\mathbf{A}^{-1} = (\det\mathbf{A})^{-1}\operatorname{adj}\mathbf{A}\,; \quad \mathbf{A}^{-1} \text{ exists if } \det\mathbf{A} \neq 0$$

Rules for the cofactor, the determinant and the inverse tensor:

$$\begin{aligned}
\det(\mathbf{A}\,\mathbf{B}) &= \det\mathbf{A}\det\mathbf{B}\\
\det(\alpha\,\mathbf{A}) &= \alpha^3\det\mathbf{A}\\
\det\mathbf{I} &= 1\\
\det\mathbf{A}^T &= \det\mathbf{A}\\
\det\overset{+}{\mathbf{A}} &= (\det\mathbf{A})^2\\
\det\mathbf{A}^{-1} &= (\det\mathbf{A})^{-1}\\
\det(\mathbf{A}+\mathbf{B}) &= \det\mathbf{A} + \overset{+}{\mathbf{A}}\cdot\mathbf{B} + \mathbf{A}\cdot\overset{+}{\mathbf{B}} + \det\mathbf{B}\\
(\mathbf{A}\,\mathbf{B})^{+} &= \overset{+}{\mathbf{A}}\,\overset{+}{\mathbf{B}}\\
(\overset{+}{\mathbf{A}})^T &= (\mathbf{A}^T)^{+}
\end{aligned}$$

2.9 The eigenvalue problem and the invariants of tensors

Definition: The eigenvalue problem of an arbitrary 2nd order tensor $\mathbf{A}$ is given by

$$(\mathbf{A} - \gamma_{\mathbf{A}}\,\mathbf{I})\,\mathbf{a} = \mathbf{0}\,, \quad \text{where} \begin{cases} \gamma_{\mathbf{A}} & : \text{eigenvalue}\\ \mathbf{a} & : \text{eigenvector}\end{cases}$$

Formal solution for $\mathbf{a}$ yields

$$\mathbf{a} = (\mathbf{A} - \gamma_{\mathbf{A}}\,\mathbf{I})^{-1}\,\mathbf{0} = \text{adj}\,(\mathbf{A} - \gamma_{\mathbf{A}}\,\mathbf{I})\,\frac{\mathbf{0}}{\det\,(\mathbf{A} - \gamma_{\mathbf{A}}\,\mathbf{I})}$$

Consequence: Non-trivial solution for $\mathbf{a}$ only if the characteristic equation is fulfilled, i. e.

$$\det\,(\mathbf{A} - \gamma_{\mathbf{A}}\,\mathbf{I}) = 0$$

With the determinant rule

$$\begin{aligned}
\det\,(\mathbf{A}+\mathbf{B}) &= \tfrac{1}{6}\,[(\mathbf{A}+\mathbf{B}) \divideontimes (\mathbf{A}+\mathbf{B})] \cdot (\mathbf{A}+\mathbf{B}) \\
&= \tfrac{1}{6}\,(\mathbf{A} \divideontimes \mathbf{A}) \cdot \mathbf{A} + \tfrac{1}{6}\,(\mathbf{A} \divideontimes \mathbf{A}) \cdot \mathbf{B} + \tfrac{1}{3}\,(\mathbf{A} \divideontimes \mathbf{B}) \cdot \mathbf{A} + \\
&\quad + \tfrac{1}{3}\,(\mathbf{A} \divideontimes \mathbf{B}) \cdot \mathbf{B} + \tfrac{1}{6}\,(\mathbf{B} \divideontimes \mathbf{B}) \cdot \mathbf{A} + \tfrac{1}{6}\,(\mathbf{B} \divideontimes \mathbf{B}) \cdot \mathbf{B} \\
&= \det\mathbf{A} + \overset{+}{\mathbf{A}} \cdot \mathbf{B} + \mathbf{A} \cdot \overset{+}{\mathbf{B}} + \det\mathbf{B}
\end{aligned}$$

follows

$$\begin{aligned}
\det\,(\mathbf{A} - \gamma_{\mathbf{A}}\,\mathbf{I}) &= \det\mathbf{A} + \overset{+}{\mathbf{A}} \cdot (-\gamma_{\mathbf{A}}\,\mathbf{I}) + \mathbf{A} \cdot (-\gamma_{\mathbf{A}}\,\mathbf{I})^{+} + \det\,(-\gamma_{\mathbf{A}}\,\mathbf{I}) \\
&= \det\mathbf{A} - \gamma_{\mathbf{A}}\,\tfrac{1}{2}\,(\mathbf{A} \divideontimes \mathbf{A}) \cdot \mathbf{I} + \gamma_{\mathbf{A}}^2\,\tfrac{1}{2}\,\mathbf{A} \cdot (\mathbf{I} \divideontimes \mathbf{I}) - \gamma_{\mathbf{A}}^3\,\det\mathbf{I} = 0
\end{aligned}$$

With the abreviations

$$\begin{aligned}
I_{\mathbf{A}} &= \tfrac{1}{2}\,(\mathbf{A} \divideontimes \mathbf{I}) \cdot \mathbf{I} \\
II_{\mathbf{A}} &= \tfrac{1}{2}\,(\mathbf{A} \divideontimes \mathbf{A}) \cdot \mathbf{I} \\
III_{\mathbf{A}} &= \tfrac{1}{6}\,(\mathbf{A} \divideontimes \mathbf{A}) \cdot \mathbf{A}
\end{aligned}$$

the characteristic equation can be simplified to

$$\det\,(\mathbf{A} - \gamma_{\mathbf{A}}\,\mathbf{I}) = III_{\mathbf{A}} - \gamma_{\mathbf{A}}\,II_{\mathbf{A}} + \gamma_{\mathbf{A}}^2\,I_{\mathbf{A}} - \gamma_{\mathbf{A}}^3 = 0$$

Rem.: The abreviations $I_{\mathbf{A}}$, $II_{\mathbf{A}}$ and $III_{\mathbf{A}}$ are the *three scalar principal invariants* of a tensor $\mathbf{A}$ which play an important role in the field of continuum mechanics.

Alternative representations of the principal inavriants

Scalar product representation:

$$\begin{aligned}
I_{\mathbf{A}} &= \mathbf{A} \cdot \mathbf{I} = \text{tr}\,\mathbf{A} \\
II_{\mathbf{A}} &= \tfrac{1}{2}\,(I_{\mathbf{A}}^2 - \mathbf{A}\,\mathbf{A} \cdot \mathbf{I}) = \tfrac{1}{2}\,[(\text{tr}\,\mathbf{A})^2 - \text{tr}\,(\mathbf{A}\,\mathbf{A})] \\
III_{\mathbf{A}} &= \tfrac{1}{6}\,I_{\mathbf{A}}^3 - \tfrac{1}{2}\,I_{\mathbf{A}}^2\,(\mathbf{A}\,\mathbf{A} \cdot \mathbf{I}) + \tfrac{1}{3}\,\mathbf{A}^T\mathbf{A}^T \cdot \mathbf{A} = \\
&= \tfrac{1}{6}\,[(\text{tr}\,\mathbf{A})^3 - 3\,\text{tr}\,\mathbf{A}\,\text{tr}\,(\mathbf{A}\,\mathbf{A}) + 2\,\text{tr}\,(\mathbf{A}\,\mathbf{A}\,\mathbf{A})] = \det\mathbf{A}
\end{aligned}$$

Eigenvalue representation:

$$\begin{aligned}
I_{\mathbf{A}} &= \gamma_{\mathbf{A}(1)} + \gamma_{\mathbf{A}(2)} + \gamma_{\mathbf{A}(3)} \\
II_{\mathbf{A}} &= \gamma_{\mathbf{A}(1)}\,\gamma_{\mathbf{A}(2)} + \gamma_{\mathbf{A}(2)}\,\gamma_{\mathbf{A}(3)} + \gamma_{\mathbf{A}(3)}\,\gamma_{\mathbf{A}(1)} \\
III_{\mathbf{A}} &= \gamma_{\mathbf{A}(1)}\,\gamma_{\mathbf{A}(2)}\,\gamma_{\mathbf{A}(3)}
\end{aligned}$$

Caley-Hamilton-Theorem:

$$\mathbf{A}\mathbf{A}\mathbf{A} - I_{\mathbf{A}}\,\mathbf{A}\mathbf{A} + II_{\mathbf{A}}\,\mathbf{A} - III_{\mathbf{A}}\,\mathbf{I} = \mathbf{0}$$

3 Fundamentals of vector and tensor analysis

3.1 Introduction of functions

Notation:

$$\text{exists} \left\{ \begin{array}{ll} \phi(\,\cdot\,) & : \text{scalar-valued function} \\ \mathbf{v}(\,\cdot\,) & : \text{vector-valued function} \\ \mathbf{T}(\,\cdot\,) & : \text{tensor-valued function} \end{array} \right\} \text{ of } (\,\cdot\,) \left\{ \begin{array}{l} \text{scalar variables} \\ \text{vector variables} \\ \text{tensor variables} \end{array} \right.$$

Example: $\phi(\mathbf{A})$: scalar-valued tensor function

Notions:

- **Domain** of a function: set of all possible values of the independent variable quantities (variables); usually contiguous
- **Range** of a function: set of all possible values of the dependent variable quantities: $\phi(\,\cdot\,)$; $\mathbf{v}(\,\cdot\,)$; $\mathbf{T}(\,\cdot\,)$

3.2 Functions of scalar variables

here: Vector- and tensor-valued functions of real scalar variables

(a) Vector-valued functions of a single variable

It exists:

$$\mathbf{u} = \mathbf{u}(\alpha) \quad \text{with} \quad \left\{ \begin{array}{ll} \mathbf{u} & : \text{unique vector-valued function,} \\ & \text{ range in the open domain } \mathcal{V}^3 \\ \alpha & : \text{real scalar variable} \end{array} \right.$$

Derivative of $\mathbf{u}(\alpha)$ with the differential quotient:

$$\mathbf{w}(\alpha) := \mathbf{u}'(\alpha) := \frac{\mathrm{d}\mathbf{u}(\alpha)}{\mathrm{d}\alpha}$$

Differential of $\mathbf{u}(\alpha)$:

$$\mathrm{d}\mathbf{u} = \mathbf{u}'(\alpha)\,\mathrm{d}\alpha$$

Introduction of higher derivatives and differentials:

$$\mathrm{d}^2\mathbf{u} = \mathrm{d}(\mathrm{d}\mathbf{u}) = \mathbf{u}''(\alpha)\,\mathrm{d}\alpha^2 = \frac{\mathrm{d}^2\mathbf{u}(\alpha)}{\mathrm{d}\alpha^2}\,\mathrm{d}\alpha^2 \quad \text{etc.}$$

(b) Vector-valued functions of several variables

It exists:

$$\mathbf{u} = \mathbf{u}(\alpha,\, \beta,\, \gamma,\, ...) \quad \text{with} \quad \{\alpha,\, \beta,\, \gamma,\, ...\} \;:\; \text{real scalar variable}$$

Partial derivative of $\mathbf{u}(\alpha,\, \beta,\, \gamma,\, ...)$:

$$\mathbf{w}_\alpha(\alpha,\, \beta,\, \gamma,\, ...) := \frac{\partial \mathbf{u}(\,\cdot\,)}{\partial \alpha} =: \mathbf{u}_{,\alpha}$$

Total differential of $\mathbf{u}(\alpha,\, \beta,\, \gamma,\, ...)$:

$$\mathrm{d}\mathbf{u} = \mathbf{u}_{,\alpha}\; \mathrm{d}\alpha + \mathbf{u}_{,\beta}\; \mathrm{d}\beta + \mathbf{u}_{,\gamma}\; \mathrm{d}\gamma + \;\cdots$$

Higher partial derivative (examples):

$$\mathbf{u}_{,\alpha\alpha} = \frac{\partial^2 \mathbf{u}(\,\cdot\,)}{\partial \alpha^2}\;; \quad \mathbf{u}_{,\gamma\beta} = \frac{\partial^2 \mathbf{u}(\,\cdot\,)}{\partial \gamma\, \partial \beta}$$

Rem.: The order of partial derivatives is permutable.

(c) Tensor functions of a single or of several variables

Rem.: Tensor-valued functions are treated analogously to the above procedure.

(d) Derivative of products of functions

Some rules:

$$\begin{aligned}(\mathbf{a} \otimes \mathbf{b})' &= \mathbf{a}' \otimes \mathbf{b} + \mathbf{a} \otimes \mathbf{b}' \\ (\mathbf{A}\,\mathbf{B})' &= \mathbf{A}'\,\mathbf{B} + \mathbf{A}\,\mathbf{B}' \\ (\mathbf{A}^{-1})' &= -\mathbf{A}^{-1}\,\mathbf{A}'\,\mathbf{A}^{-1}\end{aligned}$$

3.3 Functions of vector and tensor variables

(a) The gradient operator

Rem.: Functions of the position (placement) vector are called **field functions**. Derivatives with respect to the position vector are called “gradient of a function”.

Scalar-valued functions $\phi(\mathbf{x})$

$$\operatorname{grad} \phi(\mathbf{x}) := \frac{\mathrm{d}\phi(\mathbf{x})}{\mathrm{d}\mathbf{x}} =: \mathbf{w}(\mathbf{x}) \quad \longrightarrow \quad \text{result is a vector field}$$

or in basis notation

$$\operatorname{grad} \phi(\mathbf{x}) := \frac{\partial \phi(\mathbf{x})}{\partial x_i}\, \mathbf{e}_i =: \phi_{,i}\; \mathbf{e}_i$$

Vector-valued functions $\mathbf{v}(\mathbf{x})$

$$\operatorname{grad}\mathbf{v}(\mathbf{x}) := \frac{\mathrm{d}\mathbf{v}(\mathbf{x})}{\mathrm{d}\mathbf{x}} =: \mathbf{S}(\mathbf{x}) \quad \longrightarrow \quad \text{result is a tensor field}$$

or in basis notation

$$\operatorname{grad}\mathbf{v}(\mathbf{x}) := \frac{\partial v_i(\mathbf{x})}{\partial x_j}\,\mathbf{e}_i \otimes \mathbf{e}_j =: v_{i},_{j}\;\mathbf{e}_i \otimes \mathbf{e}_j$$

Tensor-valued functions $\mathbf{T}(\mathbf{x})$

$$\operatorname{grad}\mathbf{T}(\mathbf{x}) := \frac{\mathrm{d}\mathbf{T}(\mathbf{x})}{\mathrm{d}\mathbf{x}} =: \overset{3}{\mathbf{U}}(\mathbf{x}) \quad \longrightarrow \quad \text{result is a tensor field of 3-rd order}$$

or in basis notation

$$\operatorname{grad}\mathbf{T}(\mathbf{x}) := \frac{\partial t_{ik}(\mathbf{x})}{\partial x_j}\,\mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_j =: t_{ik},_{j}\;\mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_j$$

Rem.: The gradient operator $\operatorname{grad}(\,\cdot\,) = \nabla(\,\cdot\,)$ (with ∇ : Nabla operator) increases the order of the respective function by one.

(b) Derivative of functions of arbitrary vectorial and tensorial variables

Rem.: Derivatives concerning the respective variables are built analogously to the preceding procedures, e. g.

$$\frac{\partial\mathbf{R}(\mathbf{T},\,\mathbf{v})}{\partial\mathbf{T}} = \frac{\partial R_{ij}(\mathbf{T},\,\mathbf{v})}{\partial t_{st}}\,\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_s \otimes \mathbf{e}_t$$

Some specific rules for the derivative of tensor functions with respect to tensors

For arbitrary 2-nd order tensors $\mathbf{A}, \mathbf{B}, \mathbf{C}$, the following rules hold:

$$\begin{aligned}
\frac{\partial(\mathbf{A}\mathbf{B})}{\partial\mathbf{B}} &= (\mathbf{A}\otimes\mathbf{I})^{\overset{23}{T}} \\
\frac{\partial(\mathbf{A}\mathbf{B})}{\partial\mathbf{A}} &= (\mathbf{I}\otimes\mathbf{B}^T)^{\overset{23}{T}} \\
\frac{\partial(\mathbf{A}\mathbf{A})}{\partial\mathbf{A}} &= (\mathbf{A}\otimes\mathbf{I})^{\overset{23}{T}} + (\mathbf{I}\otimes\mathbf{A}^T)^{\overset{23}{T}} \\
\frac{\partial(\mathbf{A}^T\mathbf{A})}{\partial\mathbf{A}} &= (\mathbf{A}^T\otimes\mathbf{I})^{\overset{23}{T}} + (\mathbf{I}\otimes\mathbf{A})^{\overset{24}{T}} \\
\frac{\partial(\mathbf{A}\mathbf{A}^T)}{\partial\mathbf{A}} &= (\mathbf{A}\otimes\mathbf{I})^{\overset{24}{T}} + (\mathbf{I}\otimes\mathbf{A})^{\overset{23}{T}} \\
\frac{\partial(\mathbf{A}^T\mathbf{A}^T)}{\partial\mathbf{A}} &= (\mathbf{I}\otimes\mathbf{A}^T)^{\overset{24}{T}} + (\mathbf{A}^T\otimes\mathbf{I})^{\overset{24}{T}}
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial(\mathbf{A}\mathbf{B}\mathbf{C})}{\partial\mathbf{B}} &= (\mathbf{A}\otimes\mathbf{C}^T)^{\overset{23}{T}} \\
\frac{\partial\mathbf{A}^T}{\partial\mathbf{A}} &= (\mathbf{I}\otimes\mathbf{I})^{\overset{24}{T}} \\
\frac{\partial\mathbf{A}^{-1}}{\partial\mathbf{A}} &= -(\mathbf{A}^{-1}\otimes\mathbf{A}^{T-1})^{\overset{23}{T}} \\
\frac{\partial\mathbf{A}^{T-1}}{\partial\mathbf{A}} &= -(\mathbf{A}^{T-1}\otimes\mathbf{A}^{T-1})^{\overset{24}{T}} \\
\frac{\partial\overset{+}{\mathbf{A}}}{\partial\mathbf{A}} &= \det\mathbf{A}\,[(\mathbf{A}^{T-1}\otimes\mathbf{A}^{T-1}) - (\mathbf{A}^{T-1}\otimes\mathbf{A}^{T-1})^{\overset{24}{T}}] \\
\frac{\partial(\alpha\,\beta)}{\partial\mathbf{C}} &= \alpha\,\frac{\partial\beta}{\partial\mathbf{C}} + \beta\,\frac{\partial\alpha}{\partial\mathbf{C}} \\
\frac{\partial(\alpha\,\mathbf{v})}{\partial\mathbf{C}} &= \mathbf{v}\otimes\frac{\partial\alpha}{\partial\mathbf{C}} + \alpha\,\frac{\partial\mathbf{v}}{\partial\mathbf{C}} \\
\frac{\partial(\alpha\,\mathbf{A})}{\partial\mathbf{C}} &= \mathbf{A}\otimes\frac{\partial\alpha}{\partial\mathbf{C}} + \alpha\,\frac{\partial\mathbf{A}}{\partial\mathbf{C}} \\
\frac{\partial(\mathbf{A}\,\mathbf{v})}{\partial\mathbf{C}} &= \left[\left(\frac{\partial\mathbf{A}}{\partial\mathbf{C}}\right)^{\overset{24}{T}}\right]^{\overset{23}{T}}\mathbf{v} + \left[\mathbf{A}\,\frac{\partial\mathbf{v}}{\partial\mathbf{C}}\right]^{\underline{3}} \\
\frac{\partial(\mathbf{u}\cdot\mathbf{v})}{\partial\mathbf{C}} &= \left[\left(\frac{\partial\mathbf{u}}{\partial\mathbf{C}}\right)^{\overset{13}{T}}\mathbf{v}\right]^{T} + \left[\left(\frac{\partial\mathbf{v}}{\partial\mathbf{C}}\right)^{\overset{13}{T}}\mathbf{u}\right]^{T} \\
\frac{\partial(\mathbf{A}\cdot\mathbf{B})}{\partial\mathbf{C}} &= \left(\frac{\partial\mathbf{A}}{\partial\mathbf{C}}\right)^{T}\mathbf{B} + \left(\frac{\partial\mathbf{B}}{\partial\mathbf{C}}\right)^{T}\mathbf{A} \\
\frac{\partial(\mathbf{A}\mathbf{B})}{\partial\mathbf{C}} &= \left(\left[\left(\frac{\partial\mathbf{A}}{\partial\mathbf{C}}\right)^{\overset{24}{T}}\mathbf{B}\right]^{\underline{4}}\right)^{\overset{24}{T}} + \left(\left[\left(\frac{\partial\mathbf{B}}{\partial\mathbf{C}}\right)^{\overset{14}{T}}\mathbf{A}^{T}\right]^{\underline{4}}\right)^{\overset{14}{T}}
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\frac{\partial\mathbf{A}}{\partial\mathbf{A}} &= (\mathbf{I}\otimes\mathbf{I})^{\overset{23}{T}} =: \overset{4}{\mathbf{I}} \\
\frac{\partial\mathbf{A}^T}{\partial\mathbf{A}} &= (\mathbf{I}\otimes\mathbf{I})^{\overset{24}{T}} \\
\frac{\partial(\mathbf{A}\cdot\mathbf{I})\,\mathbf{I}}{\partial\mathbf{A}} &= (\mathbf{I}\otimes\mathbf{I}) \\
\frac{\partial\,\overset{\mathrm{A}}{\mathbf{t}}(\mathbf{A})}{\partial\mathbf{A}} &= -\tfrac{1}{2}\ \overset{3}{\mathbf{E}}
\end{aligned}
$$

Principal invariants and their derivatives (see also section 2.9)

$$
\begin{aligned}
\frac{\partial I_{\mathbf{A}}}{\partial\mathbf{A}} &= \mathbf{I} && \text{with} && I_{\mathbf{A}} = \mathbf{A}\cdot\mathbf{I} \\
\frac{\partial II_{\mathbf{A}}}{\partial\mathbf{A}} &= \mathbf{A}\divideontimes\mathbf{I} && \text{with} && II_{\mathbf{A}} = \tfrac{1}{2}\,(I_{\mathbf{A}}^2 - \mathbf{A}\,\mathbf{A}\cdot\mathbf{I}) \\
\frac{\partial III_{\mathbf{A}}}{\partial\mathbf{A}} &= \overset{+}{\mathbf{A}} && \text{with} && III_{\mathbf{A}} = \det\mathbf{A}
\end{aligned}
$$

(c) Specific operators

here: Introduction of the further differential operators div (·) and rot (·).

Divergence of a vector field v(x)

$$\operatorname{div}\mathbf{v}(\mathbf{x}) := \operatorname{grad}\mathbf{v}(\mathbf{x})\cdot\mathbf{I} =: \phi(\mathbf{x}) \quad\longrightarrow\quad \text{result is a scalar field}$$

or in basis notation

$$\begin{aligned}\operatorname{div}\mathbf{v}(\mathbf{x}) &= v_{i},_{j}\;(\mathbf{e}_i\otimes\mathbf{e}_j)\cdot(\mathbf{e}_n\otimes\mathbf{e}_n)\\ &= v_{i},_{j}\;\delta_{in}\,\delta_{jn} = v_{n},_{n}\\ &= \frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}\end{aligned}$$

Divergence of a tensor field T(x)

$$\operatorname{div}\mathbf{T}(\mathbf{x}) = [\operatorname{grad}\mathbf{T}(\mathbf{x})]\,\mathbf{I} =: \mathbf{v}(\mathbf{x}) \quad\longrightarrow\quad \text{result is a vector field}$$

or in basis notation

$$\begin{aligned}\operatorname{div}\mathbf{T}(\mathbf{x}) &= t_{ik},_{j}\;(\mathbf{e}_i\otimes\mathbf{e}_k\otimes\mathbf{e}_j)\,(\mathbf{e}_n\otimes\mathbf{e}_n)\\ &= t_{ik},_{j}\;\delta_{kn}\,\delta_{jn}\,\mathbf{e}_i = t_{in},_{n}\;\mathbf{e}_i\end{aligned}$$

Rem.: The divergence operator div (·) = ∇ · (·) decreases the order of the respective function by one.

Rotation of a vector field v(x)

$$\operatorname{rot}\mathbf{v}(\mathbf{x}) := \overset{3}{\mathbf{E}}\,[\operatorname{grad}\mathbf{v}(\mathbf{x})]^T =: \mathbf{r}(\mathbf{x}) \quad\longrightarrow\quad \text{result is a vector field}$$

or in basis notation

$$\begin{aligned}\operatorname{rot}\mathbf{v}(\mathbf{x}) &= e_{ijn}\,(\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_n)\,v_{o},_{p}\;(\mathbf{e}_p\otimes\mathbf{e}_o)\\ &= e_{ijn}\,v_{o},_{p}\;\delta_{jp}\,\delta_{no}\,\mathbf{e}_i = e_{ijn}\,v_{n},_{j}\;\mathbf{e}_i\end{aligned}$$

Consequence: rot **v**(**x**) yields twice the axial vector corresponding to the skew-symmetric part of grad **v**(**x**).

Rem.: The rotation operator rot (·) = curl (·) = ∇ × (·) preserves the order of the respective function.

Laplace **operator**

$$\Delta(\,\cdot\,) := \operatorname{div}\operatorname{grad}(\,\cdot\,) \;\longrightarrow\; \text{analogical to the precedings}$$

Rem.: The Laplace operator Δ(·) = ∇·∇(·) preserves the order of the differentiated function.

Rules for the operators grad (·), div (·), and rot (·)

$$
\begin{aligned}
\operatorname{grad}(\phi\psi) &= \phi \operatorname{grad}\psi + \psi \operatorname{grad}\phi \\
\operatorname{grad}(\phi\mathbf{v}) &= \mathbf{v} \otimes \operatorname{grad}\phi + \phi \operatorname{grad}\mathbf{v} \\
\operatorname{grad}(\phi\mathbf{T}) &= \mathbf{T} \otimes \operatorname{grad}\phi + \phi \operatorname{grad}\mathbf{T} \\
\operatorname{grad}(\mathbf{u}\cdot\mathbf{v}) &= (\operatorname{grad}\mathbf{u})^T\,\mathbf{v} + (\operatorname{grad}\mathbf{v})^T\,\mathbf{u} \\
\operatorname{grad}(\mathbf{u}\times\mathbf{v}) &= \mathbf{u} \times \operatorname{grad}\mathbf{v} + \operatorname{grad}\mathbf{u} \times \mathbf{v} \\
\operatorname{grad}(\mathbf{a}\otimes\mathbf{b}) &= [\operatorname{grad}\mathbf{a} \otimes \mathbf{b} + \mathbf{a} \otimes (\operatorname{grad}\mathbf{b})^T]^{\overset{23}{T}} \\
\operatorname{grad}(\mathbf{T}\mathbf{v}) &= (\operatorname{grad}\mathbf{T})^{\overset{23}{T}}\mathbf{v} + \mathbf{T} \operatorname{grad}\mathbf{v} \\
\operatorname{grad}(\mathbf{T}\mathbf{S}) &= [(\operatorname{grad}\mathbf{T})^{\overset{23}{T}}\mathbf{S}]^{\underline{3}\overset{23}{T}} + (\mathbf{T} \operatorname{grad}\mathbf{S})^{\underline{3}} \\
\operatorname{grad}(\mathbf{T}\cdot\mathbf{S}) &= (\operatorname{grad}\mathbf{T})^{\overset{13}{T}}\,\mathbf{S}^T + (\operatorname{grad}\mathbf{S})^{\overset{13}{T}}\,\mathbf{T}^T \\
\operatorname{grad}\mathbf{x} &= \mathbf{I} \\
\operatorname{div}(\mathbf{u}\otimes\mathbf{v}) &= \mathbf{u} \operatorname{div}\mathbf{v} + (\operatorname{grad}\mathbf{u})\,\mathbf{v} \\
\operatorname{div}(\phi\,\mathbf{v}) &= \mathbf{v} \cdot \operatorname{grad}\phi + \phi \operatorname{div}\mathbf{v} \\
\operatorname{div}(\mathbf{T}\mathbf{v}) &= (\operatorname{div}\mathbf{T}^T) \cdot \mathbf{v} + \mathbf{T}^T \cdot \operatorname{grad}\mathbf{v} \\
\operatorname{div}(\operatorname{grad}\mathbf{v})^T &= \operatorname{grad}\operatorname{div}\mathbf{v} \\
\operatorname{div}(\mathbf{u}\times\mathbf{v}) &= (\operatorname{grad}\mathbf{u} \times \mathbf{v}) \cdot \mathbf{I} - (\operatorname{grad}\mathbf{v} \times \mathbf{u}) \cdot \mathbf{I} \\
&= \mathbf{v} \cdot \operatorname{rot}\mathbf{u} - \mathbf{u} \cdot \operatorname{rot}\mathbf{v} \\
\operatorname{div}(\phi\,\mathbf{T}) &= \mathbf{T} \operatorname{grad}\phi + \phi \operatorname{div}\mathbf{T} \\
\operatorname{div}(\mathbf{T}\,\mathbf{S}) &= (\operatorname{grad}\mathbf{T})\,\mathbf{S} + \mathbf{T} \operatorname{div}\mathbf{S} \\
\operatorname{div}(\mathbf{v}\times\mathbf{T}) &= \mathbf{v} \times \operatorname{div}\mathbf{T} + \operatorname{grad}\mathbf{v} \times \mathbf{T} \\
\operatorname{div}(\mathbf{v}\otimes\mathbf{T}) &= \mathbf{v} \otimes \operatorname{div}\mathbf{T} + (\operatorname{grad}\mathbf{v})\,\mathbf{T}^T \\
\operatorname{div}(\mathbf{v}\otimes\overset{3}{\mathbf{T}}) &= \mathbf{v} \otimes \operatorname{div}\overset{3}{\mathbf{T}} + [(\operatorname{grad}\mathbf{v})\,(\,\overset{3}{\mathbf{T}}{}^{\overset{13}{T}})^{\overset{23}{T}}]^{\underline{3}} \\
\operatorname{div}(\operatorname{grad}\mathbf{v})^+ &= \mathbf{0} \\
\operatorname{div}[\operatorname{grad}\mathbf{v} \pm (\operatorname{grad}\mathbf{v})^T] &= \operatorname{div}\operatorname{grad}\mathbf{v} \pm \operatorname{grad}\operatorname{div}\mathbf{v} \\
\operatorname{div}\operatorname{rot}\mathbf{v} &= 0 \\
\operatorname{rot}\operatorname{rot}\mathbf{v} &= \operatorname{grad}\operatorname{div}\mathbf{v} - \operatorname{div}\operatorname{grad}\mathbf{v} \\
\operatorname{rot}\operatorname{grad}\phi &= \mathbf{0}
\end{aligned}
$$

$$\begin{aligned}
\operatorname{rot}\operatorname{grad}\mathbf{v} &= \mathbf{0}\\
\operatorname{rot}(\operatorname{grad}\mathbf{v})^T &= \operatorname{grad}\operatorname{rot}\mathbf{v}\\
\operatorname{rot}(\phi\,\mathbf{v}) &= \phi\operatorname{rot}\mathbf{v} + \operatorname{grad}\phi\times\mathbf{v}\\
\operatorname{rot}(\mathbf{u}\times\mathbf{v}) &= \operatorname{div}(\mathbf{u}\otimes\mathbf{v} - \mathbf{v}\otimes\mathbf{u})\\
&= \mathbf{u}\operatorname{div}\mathbf{v} + (\operatorname{grad}\mathbf{u})\mathbf{v} - \mathbf{v}\operatorname{div}\mathbf{u} - (\operatorname{grad}\mathbf{v})\mathbf{u}
\end{aligned}$$

GRASSMANN **evolution:**

$$\mathbf{v}\times\operatorname{rot}\mathbf{v} = \tfrac{1}{2}\operatorname{grad}(\mathbf{v}\cdot\mathbf{v}) - (\operatorname{grad}\mathbf{v})\,\mathbf{v} = (\operatorname{grad}\mathbf{v})^T\mathbf{v} - (\operatorname{grad}\mathbf{v})\,\mathbf{v}$$

3.4 Integral theorems

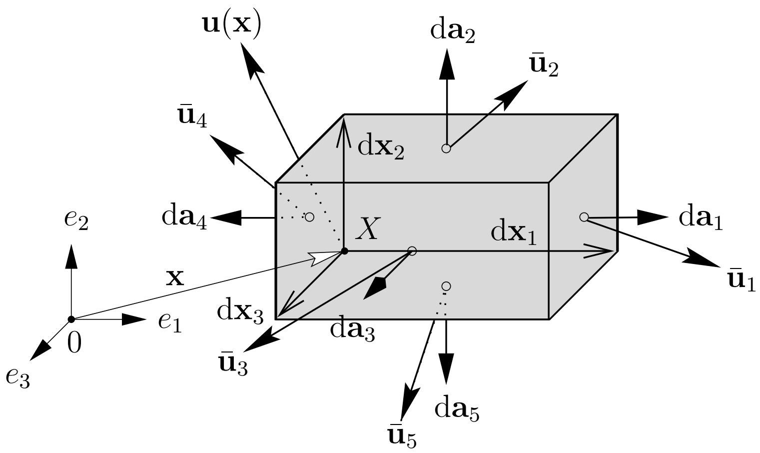
Rem.: In what follows, some integral theorems for the transformation of surface integrals into volume integrals are presented.

Requirement: $\mathbf{u} = \mathbf{u}(\mathbf{x})$ is a steady and sufficiently often steadily differentiable vector field. The domain of $\mathbf{u}$ is in $\mathcal{V}^3$.

(a) PROOF OF THE INTEGRAL THEOREM

$$\int_{\mathcal{S}} \mathbf{u}(\mathbf{x})\otimes \mathrm{d}\mathbf{a} = \int_V \operatorname{grad}\mathbf{u}(\mathbf{x})\mathrm{d}v \qquad \text{with } \mathrm{d}\mathbf{a} = \mathbf{n}\,\mathrm{d}a$$

and $\left\{\begin{array}{ll} \mathrm{d}a & : \text{surface element}\\ \mathbf{n} & : \text{outward oriented unit surface normal vector}\end{array}\right.$



Basis: Consideration of an infinitesimal volume element $\mathrm{d}v$ spanned in the point X by the position vector $\mathbf{x}$, and $\bar{\mathbf{u}}_i$, i. e. the values of $\mathbf{u}(\mathbf{x})$ in the centroid of the partial surfaces 1 - 6.

Determination of the surface element vectors $\mathrm{d}\mathbf{a}_i$**:**

$$\begin{aligned}\mathrm{d}\mathbf{a}_1 &= \mathrm{d}\mathbf{x}_2 \times \mathrm{d}\mathbf{x}_3 = \mathrm{d}x_2\,\mathrm{d}x_3\,(\mathbf{e}_2 \times \mathbf{e}_3) \\ &= \mathrm{d}x_2\,\mathrm{d}x_3\,\mathbf{e}_1 = -\mathrm{d}\mathbf{a}_4 \;\longrightarrow\; \mathbf{e}_1 = \mathbf{n}_1 = -\mathbf{n}_4\end{aligned}$$

Furthermore, one obtains

$$\begin{aligned}\mathrm{d}\mathbf{a}_2 &= \mathrm{d}x_3\,\mathrm{d}x_1\,\mathbf{e}_2 = -\mathrm{d}\mathbf{a}_5 \;\longrightarrow\; \mathbf{e}_2 = \mathbf{n}_2 = -\mathbf{n}_5 \\ \mathrm{d}\mathbf{a}_3 &= \mathrm{d}x_1\,\mathrm{d}x_2\,\mathbf{e}_3 = -\mathrm{d}\mathbf{a}_6 \;\longrightarrow\; \mathbf{e}_3 = \mathbf{n}_3 = -\mathbf{n}_6\end{aligned}$$

Rem.: The surface vectors hold the condition $\displaystyle\sum_{i=1}^{6} \mathrm{d}\mathbf{a}_i = \mathbf{0}$.

Determination of the volume elements $\mathrm{d}v$**:**

$$\mathrm{d}v = (\mathrm{d}\mathbf{x}_1 \times \mathrm{d}\mathbf{x}_2) \cdot \mathrm{d}\mathbf{x}_3 = \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$$

Values of $\mathbf{u}(\mathbf{x})$ **in the centroids of the partial surfaces:**

Rem.: The increments of $\mathbf{u}(\mathbf{x})$ in the directions of $\mathrm{d}x_1$, $\mathrm{d}x_2$, $\mathrm{d}x_3$ are approximated by the first term of a TAYLOR series.

$$\begin{aligned}\bar{\mathbf{u}}_4 &= \mathbf{u}(\mathbf{x}) + \frac{1}{2}\frac{\partial \mathbf{u}}{\partial x_2}\,\mathrm{d}x_2 + \frac{1}{2}\frac{\partial \mathbf{u}}{\partial x_3}\,\mathrm{d}x_3 \\ \bar{\mathbf{u}}_1 &= \bar{\mathbf{u}}_4 + \frac{\partial \mathbf{u}}{\partial x_1}\,\mathrm{d}x_1\end{aligned}$$

Furthermore, one obtains

$$\bar{\mathbf{u}}_2 = \bar{\mathbf{u}}_5 + \frac{\partial \mathbf{u}}{\partial x_2}\,\mathrm{d}x_2\ , \quad \bar{\mathbf{u}}_3 = \bar{\mathbf{u}}_6 + \frac{\partial \mathbf{u}}{\partial x_3}\,\mathrm{d}x_3$$

Computation of the surface integral yields

$$\int\limits_{\mathcal{S}(\mathrm{d}v)} \mathbf{u}(\mathbf{x}) \otimes \mathrm{d}\mathbf{a} \;\longrightarrow\; \sum_{i=1}^{6} \bar{\mathbf{u}}_i \otimes \mathrm{d}\mathbf{a}_i = \bar{\mathbf{u}}_1 \otimes \mathrm{d}\mathbf{a}_1 + \underbrace{\bar{\mathbf{u}}_4 \otimes \mathrm{d}\mathbf{a}_4}_{(\bar{\mathbf{u}}_1 - \frac{\partial \mathbf{u}}{\partial x_1}\mathrm{d}x_1) \otimes (-\mathrm{d}\mathbf{a}_1)} + \cdots$$

Thus

$$\sum_{i=1}^{6} \bar{\mathbf{u}}_i \otimes \mathrm{d}\mathbf{a}_i = \frac{\partial \mathbf{u}}{\partial x_1}\,\mathrm{d}x_1 \otimes \mathrm{d}\mathbf{a}_1 + \frac{\partial \mathbf{u}}{\partial x_2}\,\mathrm{d}x_2 \otimes \mathrm{d}\mathbf{a}_2 + \frac{\partial \mathbf{u}}{\partial x_3}\,\mathrm{d}x_3 \otimes \mathrm{d}\mathbf{a}_3$$

with

$$\mathrm{d}\mathbf{a}_1 = \mathrm{d}x_2\,\mathrm{d}x_3\,\mathbf{e}_1\ ,\ \mathrm{d}\mathbf{a}_2 = \mathrm{d}x_1\,\mathrm{d}x_3\,\mathbf{e}_2\ ,\ \mathrm{d}\mathbf{a}_3 = \mathrm{d}x_1\,\mathrm{d}x_2\,\mathbf{e}_3$$

yields

$$\sum_{i=1}^{6} \bar{\mathbf{u}}_i \otimes \mathrm{d}\mathbf{a}_i = \underbrace{\left(\frac{\partial \mathbf{u}}{\partial x_1} \otimes \mathbf{e}_1 + \frac{\partial \mathbf{u}}{\partial x_2} \otimes \mathbf{e}_2 + \frac{\partial \mathbf{u}}{\partial x_3} \otimes \mathbf{e}_3\right)}_{\frac{\partial u_i}{\partial x_j}\,\mathbf{e}_i \otimes \mathbf{e}_j = \operatorname{grad} \mathbf{u}} \underbrace{\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3}_{\mathrm{d}v}$$

Thus

$$\sum_{i=1}^{6} \bar{\mathbf{u}}_i \otimes \mathrm{d}\mathbf{a}_i = \operatorname{grad}\mathbf{u}\,\mathrm{d}v$$

Integration over an arbitrary volume V yields

$$\int_{\mathcal{S}} \mathbf{u}(\mathbf{x}) \otimes \mathrm{d}\mathbf{a} = \int_{V} \operatorname{grad}\mathbf{u}(\mathbf{x})\,\mathrm{d}v \quad \text{q. e. d.} \qquad (*)$$

(b) Proof of the GAUSSian integral theorem

$$\int_{\mathcal{S}} \mathbf{u}(\mathbf{x}) \cdot \mathrm{d}\mathbf{a} = \int_{V} \operatorname{div}\mathbf{u}(\mathbf{x})\,\mathrm{d}v$$

Basis: Integral theorem $(*)$ after scalar multiplication with the identity tensor

$$\begin{aligned} \mathbf{I} \cdot \int_{\mathcal{S}} \mathbf{u}(\mathbf{x}) \otimes \mathrm{d}\mathbf{a} \quad &= \mathbf{I} \cdot \int_{V} \operatorname{grad}\mathbf{u}(\mathbf{x})\,\mathrm{d}v \\ \longrightarrow \quad \int_{\mathcal{S}} \underbrace{\mathbf{I} \cdot [\mathbf{u}(\mathbf{x}) \otimes \mathrm{d}\mathbf{a}]}_{\mathbf{u}(\mathbf{x}) \cdot \mathrm{d}\mathbf{a}} \quad &= \int_{\mathcal{V}} \underbrace{\mathbf{I} \cdot \operatorname{grad}\mathbf{u}(\mathbf{x})}_{\operatorname{div}\mathbf{u}(\mathbf{x})}\,\mathrm{d}v \end{aligned}$$

Thus, leading to

$$\int_{\mathcal{S}} \mathbf{u}(\mathbf{x}) \cdot \mathrm{d}\mathbf{a} = \int_{V} \operatorname{div}\mathbf{u}(\mathbf{x})\,\mathrm{d}v \qquad (**)$$

(c) Proof of the integral theorem

$$\int_{\mathcal{S}} \mathbf{T}(\mathbf{x})\,\mathrm{d}\mathbf{a} = \int_{V} \operatorname{div}\mathbf{T}(\mathbf{x})\,\mathrm{d}v$$

Basis: Scalar multiplication of the surface integral with a constant vector $\mathbf{b} \in \mathcal{V}^3$

$$\mathbf{b} \cdot \int_{\mathcal{S}} \mathbf{T}(\mathbf{x})\,\mathrm{d}\mathbf{a} = \int_{\mathcal{S}} \mathbf{b} \cdot \mathbf{T}(\mathbf{x})\,\mathrm{d}\mathbf{a} = \int_{\mathcal{S}} [\mathbf{T}^T(\mathbf{x})\,\mathbf{b}] \cdot \mathrm{d}\mathbf{a} =: \int_{\mathcal{S}} \mathbf{u}(\mathbf{x}) \cdot \mathrm{d}\mathbf{a}$$

with $\quad \mathbf{u}(\mathbf{x}) := \mathbf{T}^T(\mathbf{x})\,\mathbf{b}$

It follows with the integral theorem $(**)$

$$\mathbf{b} \cdot \int_{\mathcal{S}} \mathbf{T}(\mathbf{x})\,\mathrm{d}\mathbf{a} = \int_{V} \operatorname{div}[\mathbf{T}^T(\mathbf{x})\,\mathbf{b}]\,\mathrm{d}v$$

In particular, with $\mathbf{b} = \text{const.}$ and a divergence rule follows

$$\operatorname{div}\left[\mathbf{T}^T(\mathbf{x})\,\mathbf{b}\right] = \operatorname{div}\mathbf{T}(\mathbf{x})\cdot\mathbf{b}$$

leading to

$$\mathbf{b}\cdot\int\limits_{\mathcal{S}}\mathbf{T}(\mathbf{x})\,\mathrm{d}\mathbf{a} = \int\limits_{V}\operatorname{div}\mathbf{T}(\mathbf{x})\cdot\mathbf{b}\,\mathrm{d}v$$

Thus

$$\int\limits_{\mathcal{S}}\mathbf{T}(\mathbf{x})\,\mathrm{d}\mathbf{a} = \int\limits_{V}\operatorname{div}\mathbf{T}(\mathbf{x})\,\mathrm{d}v \quad \text{q. e. d.}$$

Rem.: At this point, no further proofs are carried out.

(d) Summary of some integral theorems

For the transformation of surface integrals into volume integrals, the following relations hold:

$$\begin{aligned}
\int\limits_{\mathcal{S}}\mathbf{u}\otimes\mathrm{d}\mathbf{a} &= \int\limits_{V}\operatorname{grad}\mathbf{u}\,\mathrm{d}v\\
\int\limits_{\mathcal{S}}\phi\,\mathrm{d}\mathbf{a} &= \int\limits_{V}\operatorname{grad}\phi\,\mathrm{d}v\\
\int\limits_{\mathcal{S}}\mathbf{u}\cdot\mathrm{d}\mathbf{a} &= \int\limits_{V}\operatorname{div}\mathbf{u}\,\mathrm{d}v
\end{aligned}$$

$$\begin{aligned}
\int\limits_{\mathcal{S}}\mathbf{u}\times\mathrm{d}\mathbf{a} &= -\int\limits_{V}\operatorname{rot}\mathbf{u}\,\mathrm{d}v\\
\int\limits_{\mathcal{S}}\mathbf{T}\,\mathrm{d}\mathbf{a} &= \int\limits_{V}\operatorname{div}\mathbf{T}\,\mathrm{d}v\\
\int\limits_{\mathcal{S}}\mathbf{u}\times\mathbf{T}\,\mathrm{d}\mathbf{a} &= \int\limits_{V}\operatorname{div}(\mathbf{u}\times\mathbf{T})\,\mathrm{d}v\\
\int\limits_{\mathcal{S}}\mathbf{u}\otimes\mathbf{T}\,\mathrm{d}\mathbf{a} &= \int\limits_{V}\operatorname{div}(\mathbf{u}\otimes\mathbf{T})\,\mathrm{d}v
\end{aligned}$$

For the transformation of line into surface integrals the following relations hold:

$$
\begin{aligned}
\oint_L \mathbf{u} \otimes \mathrm{d}\mathbf{x} &= -\int_S \operatorname{grad} \mathbf{u} \times \mathrm{d}\mathbf{a} \\
\oint_L \phi \, \mathrm{d}\mathbf{x} &= -\int_S \operatorname{grad} \phi \times \mathrm{d}\mathbf{a} \\
\oint_L \mathbf{u} \cdot \mathrm{d}\mathbf{x} &= \int_S (\operatorname{rot} \mathbf{u}) \cdot \mathrm{d}\mathbf{a} \\
\oint_L \mathbf{u} \times \mathrm{d}\mathbf{x} &= \int_S (\mathbf{I} \operatorname{div} \mathbf{u} - \operatorname{grad}^T \mathbf{u}) \, \mathrm{d}\mathbf{a} \\
\oint_L \mathbf{T} \, \mathrm{d}\mathbf{x} &= \int_S (\operatorname{rot} \mathbf{T})^T \mathrm{d}\mathbf{a}
\end{aligned}
$$

with $\mathrm{d}\mathbf{a} = \mathbf{n} \, \mathrm{d}a$

Rem.: If required, further relations of the vector and tensor calculus will be presented in the respective context. The description of non-orthogonal and non-unit basis systems was not discussed in this contribution.

3.5 Transformations between actual and reference configurations

Given are the deformation gradient $\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X}$ and arbitrary vectorial and tensorial field functions $\mathbf{v}$ and $\mathbf{A}$. Then, with

$$\text{reference configuration} \quad \left\{ \begin{array}{lcl} \operatorname{Grad}(\,\cdot\,) & = & \dfrac{\partial}{\partial\mathbf{X}}(\,\cdot\,) \\ \operatorname{Div}(\,\cdot\,) & = & [\operatorname{Grad}(\,\cdot\,)]\cdot\mathbf{I} \quad \text{or} \quad [\operatorname{Grad}(\,\cdot\,)]\,\mathbf{I} \end{array} \right.$$

$$\text{actual configuration} \quad \left\{ \begin{array}{lcl} \operatorname{grad}(\,\cdot\,) & = & \dfrac{\partial}{\partial\mathbf{x}}(\,\cdot\,) \\ \operatorname{div}(\,\cdot\,) & = & [\operatorname{grad}(\,\cdot\,)]\cdot\mathbf{I} \quad \text{or} \quad [\operatorname{grad}(\,\cdot\,)]\,\mathbf{I} \end{array} \right.$$

the following relations hold:

$$\begin{array}{lcl@{\qquad}lcl} \operatorname{Grad}\mathbf{v} & = & (\operatorname{grad}\mathbf{v})\,\mathbf{F} & \operatorname{Grad}\mathbf{A} & = & [(\operatorname{grad}\mathbf{A})\,\mathbf{F}]^{\underline{3}} \\ \operatorname{grad}\mathbf{v} & = & (\operatorname{Grad}\mathbf{v})\,\mathbf{F}^{-1} & \operatorname{grad}\mathbf{A} & = & [(\operatorname{Grad}\mathbf{A})\,\mathbf{F}^{-1}]^{\underline{3}} \\ & & & & & \\ \operatorname{Div}\mathbf{v} & = & (\operatorname{grad}\mathbf{v})\cdot\mathbf{F}^{T} & \operatorname{Div}\mathbf{A} & = & (\operatorname{grad}\mathbf{A})\,\mathbf{F}^{T} \\ \operatorname{div}\mathbf{v} & = & (\operatorname{Grad}\mathbf{v})\cdot\mathbf{F}^{T-1} & \operatorname{div}\mathbf{A} & = & (\operatorname{Grad}\mathbf{A})\,\mathbf{F}^{T-1} \end{array}$$

Furthermore, it can be shown that

$$\begin{array}{lclcl} \operatorname{Div}\mathbf{F}^{T-1} & = & -\mathbf{F}^{T-1}\,(\mathbf{F}^{T-1}\operatorname{Grad}\mathbf{F})^{\underline{1}} & = & -(\det\mathbf{F})^{-1}\,\mathbf{F}^{T-1}\,[\operatorname{Grad}(\det\mathbf{F})] \\ \operatorname{div}\mathbf{F}^{T} & = & -\mathbf{F}^{T}\,(\mathbf{F}^{T}\operatorname{grad}\mathbf{F}^{-1})^{\underline{1}} & = & -(\det\mathbf{F})\,\mathbf{F}^{T}\,[\operatorname{grad}(\det\mathbf{F})^{-1}] \end{array}$$