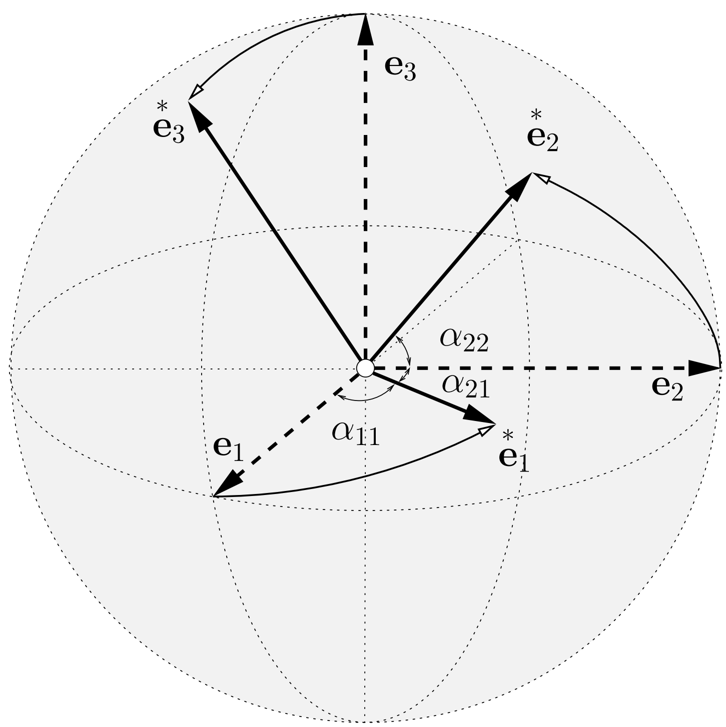
University of Stuttgart

Prof. Dr.-Ing. Dr. h. c.
Wolfgang Ehlers, Emeritus
Institute of Applied Mechanics

Vector and Tensor Calculus: A Supplement to Continuum Mechanics Research



Release of 3 February 2025

Contents

1 Mathematical Prerequisites

1.1 Basics of vector calculus

(a) Symbols, summation convention, Kronecker δ

Single- or multiple subscripts

$$
\begin{array}{lll}
u_i & \longrightarrow & u_1,\ u_2,\ u_3,\ \ldots \\
u_i\, v_k & \longrightarrow & u_1\, v_1,\ u_1\, v_2,\ u_1\, v_3,\ \ldots \\
 & & u_2\, v_1,\ u_2\, v_2,\ \ldots \\
 & & \ldots \\
t_{ik} & \longrightarrow & t_{11},\ t_{12},\ \ldots \\
 & & \ldots
\end{array}
$$

Einstein's summation convention

> **Definition:** Whenever the same subscript occurs twicely in a term, a summation over that "double" subscript has to be carried out.

Albert Einstein (1879-1955) was a German-Austrian-American theoretical physicist and a physics professor at Prag University, ETH Zürich, TU Berlin and Princeton University.

Example:

$$
\begin{aligned}
u_j\, v_j &= u_1\, v_1 + u_2\, v_2 + \ldots + u_n\, v_n \\
&= \sum_{j=1}^{n} u_j\, v_j
\end{aligned}
$$

Kronecker symbol

> **Definition:** It exists a symbol δ_{ik} with the following properties
>
> $$\delta_{ik} = \begin{cases} 0, & \text{if } i \neq k \\ 1, & \text{if } i = k \end{cases}$$

Leopold Kronecker (1823-1891) was a German mathematician who worked as a private tutor and became later a professor of mathematics at the University of Berlin.

Example:

$$
u_i\, \delta_{ik} = u_1\, \delta_{1k} + u_2\, \delta_{2k} + \ldots + u_n\, \delta_{nk}
$$

$$
\text{with} \quad u_1\, \delta_{1k} = \begin{cases} u_1\, \delta_{11} = u_1 \\ u_1\, \delta_{12} = 0 \\ \quad \vdots \\ u_1\, \delta_{1n} = 0 \end{cases}
$$

$$
\longrightarrow \quad u_i\, \delta_{ik} = u_k
$$

If the Kronecker symbol is multiplied with another quantity and if there is a double subscript in this term, the Kronecker symbol disappears, the "double" subscript can be dropped and the free subscript remains.

Remark: Subscripts occurring twicely in a term can be renamed arbitrarily.

(b) Terms and definitions of vector algebra

Remark: The following statements are related to the standard three-dimensional (3-d) physical space meaning the EUCLIDian vector space $\mathcal{V}^3$.
Generally, SPACE is a mathematical concept of a set and does not directly refer to the 3-d point space $\mathcal{E}^3$ and the 3-d vector space $\mathcal{V}^3$.

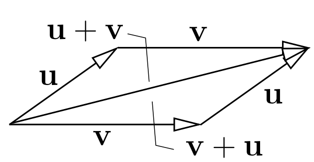
A: Vector addition

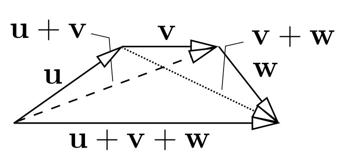
Requirement: $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, ...\} \in \mathcal{V}^3$

The following relations hold:

$$\begin{aligned} \mathbf{u}+\mathbf{v} &= \mathbf{v}+\mathbf{u} && : \text{commutative law} \\ \mathbf{u}+(\mathbf{v}+\mathbf{w}) &= (\mathbf{u}+\mathbf{v})+\mathbf{w} && : \text{associative law} \\ \mathbf{u}+\mathbf{0} &= \mathbf{u} && : \ \mathbf{0} \ : \text{identity element of vector addition} \\ \mathbf{u}+(-\mathbf{u}) &= \mathbf{0} && : \ -\mathbf{u} \ : \text{inverse element of vector addition} \end{aligned}$$

Examples to the commutative and the associative laws:





B: Multiplication of a vector with a scalar quantity

Requirement: $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, ...\} \in \mathcal{V}^3; \ \{\alpha, \beta, ...\} \in \mathbb{R}$

$$\begin{aligned} 1\,\mathbf{v} &= \mathbf{v} && : 1\text{: identity element} \\ \alpha\,(\beta\,\mathbf{v}) &= (\alpha\,\beta)\,\mathbf{v} && : \text{associative law} \\ (\alpha+\beta)\,\mathbf{v} &= \alpha\,\mathbf{v}+\beta\,\mathbf{v} && : \text{distributive law (addition of scalars)} \\ \alpha\,(\mathbf{v}+\mathbf{w}) &= \alpha\,\mathbf{v}+\alpha\,\mathbf{w} && : \text{distributive law (addition of vectors)} \\ \alpha\,\mathbf{v} &= \mathbf{v}\,\alpha && : \text{commutative law} \end{aligned}$$

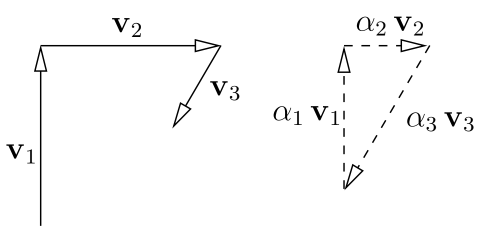
Remark: In the general vector calculus, the definitions A and B constitute the "affine vector space".

Linear dependency of vectors

Remark: In $\mathcal{V}^3$, three non-coplanar vectors are linearly independent, meaning that each further vector can be expressed as a multiple of these vectors.

> **Theorem:** The vectors $\mathbf{v}_i$ $(i = 1,\, 2,\, 3,\, ...,\, n)$ are linearly dependent, if real numbers α_i exist which are not all equal to zero, such that
>
> $$\alpha_i\,\mathbf{v}_i = \mathbf{0} \quad \text{or} \quad \alpha_1\,\mathbf{v}_1 + \alpha_2\,\mathbf{v}_2 + ... + \alpha_n\,\mathbf{v}_n = \mathbf{0}$$

Example (plane case):



$\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 \neq \mathbf{0}$

<u>but:</u> $\alpha_1\,\mathbf{v}_1 + \alpha_2\,\mathbf{v}_2 + \alpha_3\,\mathbf{v}_3 = \mathbf{0}$

$\longrightarrow$ $\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3\}$: linearly dependent

$\longrightarrow$ $\{\mathbf{v}_1,\, \mathbf{v}_2\}$: linearly independent

Remark: The α_i can be multiplied by any factor λ.

Basis vectors in $\mathcal{V}^3$

ex. : $\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3\}$: linearly independent

then : $\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \mathbf{v}\}$: linearly dependent

Thus, it follows that

$$\alpha_1\,\mathbf{v}_1 + \alpha_2\,\mathbf{v}_2 + \alpha_3\,\mathbf{v}_3 + \lambda\,\mathbf{v} = \mathbf{0}$$

$$\longrightarrow \quad \lambda\,\mathbf{v} = -\alpha_i\,\mathbf{v}_i$$

$$\text{or} \quad \mathbf{v} = \frac{-\alpha_i}{\lambda}\,\mathbf{v}_i =: \beta_i\,\mathbf{v}_i$$

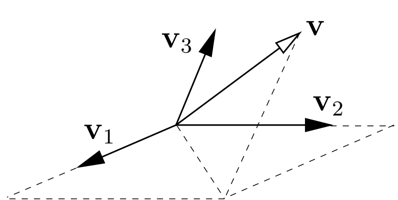
$$\text{with} \begin{cases} \beta_i = \dfrac{-\alpha_i}{\lambda} & : \quad \text{coefficients (of the vector components)} \\ \mathbf{v}_i & : \quad \text{basis vectors of } \mathbf{v} \end{cases}$$

Choice of a specific basis system

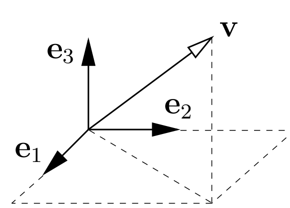
Remark: In $\mathcal{V}^3$, each system of three linearly independent vectors can be selected as a basis; e. g.

$\mathbf{v}_i$: <u>general basis</u>

$\mathbf{e}_i$: <u>specific, orthonormal basis</u> (Cartesian, right-handed)



Basissystem $\mathbf{v}_i$



Basissystem $\mathbf{e}_i$

Representation of the vector $\mathbf{v}$:

$$\mathbf{v} = \begin{cases} \beta_i \, \mathbf{v}_i \\ \gamma_i \, \mathbf{e}_i \end{cases}$$

here: Specific choice of the Cartesian basis system $\mathbf{e}_i$

Notations

$$\mathbf{v} \;=\; v_i \, \mathbf{e}_i \;=\; v_1 \, \mathbf{e}_1 + v_2 \, \mathbf{e}_2 + v_3 \, \mathbf{e}_3$$

$$\text{with} \quad \begin{cases} v_i \, \mathbf{e}_i & : \ \text{vector components} \\ v_i & : \ \text{coefficients of the vector components} \end{cases}$$

C: Scalar product of vectors

The scalar product of vectors is defined by the dot operator (dot product). The result of the product is a scalar (scalar product).

The following relations hold:

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= \mathbf{v} \cdot \mathbf{u} && : \text{commutative law} \\ \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} && : \text{distributive law} \\ \alpha \, (\mathbf{u} \cdot \mathbf{v}) &= \mathbf{u} \cdot (\alpha \, \mathbf{v}) = (\alpha \, \mathbf{u}) \cdot \mathbf{v} && : \text{associative law} \\ \mathbf{u} \cdot \mathbf{v} &= 0 \qquad \forall \, \mathbf{u}, \text{ if } \ \mathbf{v} \equiv \mathbf{0} \\ \longrightarrow \mathbf{u} \cdot \mathbf{u} &\neq 0 \qquad , \text{ if } \ \mathbf{u} \neq \mathbf{0} \end{aligned}$$

Remark: The definitions A, B and C constitute the "EUCLIDian vector space". In case that $\mathbf{u} \cdot \mathbf{u} \neq 0$, especially when

$$\mathbf{u} \cdot \mathbf{u} > 0 \ \ , \text{ if } \ \mathbf{u} \neq \mathbf{0},$$

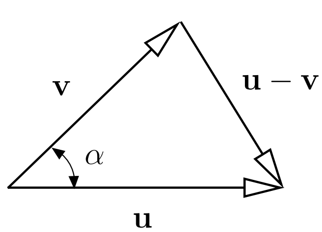
then A, B and C define the "proper EUCLIDian vector space $\mathcal{V}^3$" (physical space).

Square and norm of a vector

$$\mathbf{v}^2 := \mathbf{v} \cdot \mathbf{v} \quad , \quad v = |\mathbf{v}| = \sqrt{\mathbf{v}^2}$$

Remark: The norm is the value or the positive square root of the vector.

Angle between two vectors



$$\sphericalangle(\mathbf{u}\,;\,\mathbf{v}) =: \alpha$$

Law of cosines

$$|\mathbf{u}-\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2\,|\mathbf{u}|\,|\mathbf{v}|\,\cos\alpha$$

$$\longrightarrow \qquad \cos\alpha = \frac{\mathbf{u}^2 + \mathbf{v}^2 - (\mathbf{u}-\mathbf{v})^2}{2\,|\mathbf{u}|\,|\mathbf{v}|} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}$$

or $$\boxed{\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\,\cos\alpha}$$

Scalar product (inner product*) in an orthonormal basis

Scalar product of the basis vectors $\mathbf{e}_i$:

$$\sphericalangle(\mathbf{e}_i\,;\,\mathbf{e}_k) \begin{cases} 90°, & \text{if } i \neq k \quad : \quad \cos 90° = 0 \\ 0°, & \text{if } i = k \quad : \quad \cos 0° = 1 \end{cases}$$

thus
$$\begin{aligned} \mathbf{e}_i\cdot\mathbf{e}_k &= |\mathbf{e}_i|\,|\mathbf{e}_k|\,\cos\sphericalangle(\mathbf{e}_i\,;\,\mathbf{e}_k) \\ &= \cos\sphericalangle(\mathbf{e}_i\,;\,\mathbf{e}_k) \end{aligned}$$

It follows with the KRONECKER δ

$$\boxed{\mathbf{e}_i\cdot\mathbf{e}_k = \delta_{ik} = \begin{cases} 1, & \text{if } i = k \\ 0, & \text{if } i \neq k \end{cases}}$$

Scalar product of two vectors:

$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} &= (u_i\,\mathbf{e}_i)\cdot(v_k\,\mathbf{e}_k) \\ &= u_i\,v_k\,(\mathbf{e}_i\cdot\mathbf{e}_k) \\ &= u_i\,v_k\,\delta_{ik} \\ &= u_i\,v_i = u_1\,v_1 + u_2\,v_2 + u_3\,v_3 \end{aligned}$$

D: Vector or cross product (outer product†) of vectors

The vector product of vectors is defined by the cross operator (cross product). The result of the product is a vector (vector product).

One defines the following vector product

$$\mathbf{u}\times\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\,\sin\sphericalangle(\mathbf{u}\,;\,\mathbf{v})\,\mathbf{n}$$

with $\mathbf{n}$: unit vector $\perp \mathbf{u}\,,\ \mathbf{v}$ (corkscrew rule or right-hand rule, see page 7)

From the above definition, the following relations can be derived

$$\begin{aligned} \mathbf{u}\times\mathbf{v} &= -\mathbf{v}\times\mathbf{u} && : \underline{\text{no}}\text{ commutative law} \\ \mathbf{u}\times(\mathbf{v}+\mathbf{w}) &= \mathbf{u}\times\mathbf{v} + \mathbf{u}\times\mathbf{w} && : \text{distributive law} \\ \alpha\,(\mathbf{u}\times\mathbf{v}) &= (\alpha\,\mathbf{u})\times\mathbf{v} = \mathbf{u}\times(\alpha\,\mathbf{v}) && : \text{associative law} \end{aligned}$$

*The explanation of the notion "inner product" can be found in the Appendix on p. 51

†The explanation of the notion "outer product" can be found in the Appendix on p. 55

Scalar triple product (parallelepidial product):

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$$

Arithmetic laws for the vector product (without proof)

$$\begin{aligned} \mathbf{u} \times \mathbf{u} &= \mathbf{0} \\ (\mathbf{u} + \mathbf{v}) \times \mathbf{w} &= \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w} \\ \mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) &= \mathbf{v} \cdot (\mathbf{u} \times \mathbf{u}) = 0 \end{aligned}$$

Expansion theorem:

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\, \mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\, \mathbf{w}$$

LAGRANGEan identity:

$$(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{z}) = (\mathbf{u} \cdot \mathbf{w})\, (\mathbf{v} \cdot \mathbf{z}) - (\mathbf{u} \cdot \mathbf{z})\, (\mathbf{v} \cdot \mathbf{w})$$

Joseph-Louis LAGRANGE (1736–1813) was an Italian-French mathematician and astronomer.

Norm of the vector product:

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\, |\mathbf{v}| \sin \sphericalangle(\mathbf{u}\, ; \mathbf{v})$$

Vector product in an orthonormal basis

here: Simplified representation in matrix notation

Calculation of

$$\begin{aligned} \mathbf{u} \quad &= \quad \mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= \quad (v_2\, w_3 - v_3\, w_2)\, \mathbf{e}_1 - (v_1\, w_3 - v_3\, w_1)\, \mathbf{e}_2 + (v_1\, w_2 - v_2\, w_1)\, \mathbf{e}_3 \end{aligned}$$

Remark: If $\mathbf{u} \perp \{\mathbf{v},\, \mathbf{w}\}$, then $\mathbf{u} \cdot \mathbf{v} \ = \ \mathbf{u} \cdot \mathbf{w} \ = \ 0$

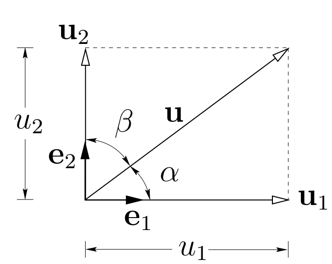
Example:

$$\mathbf{u} \cdot \mathbf{v} \ = \ u_i\, v_i \ = \ (v_2\, w_3 - v_3\, w_2)\, v_1 - (v_1\, w_3 - v_3\, w_1)\, v_2 + (v_1\, w_2 - v_2\, w_1)\, v_3 \ = \ 0 \quad \text{q. e. d.}$$

Remarks on the products between vectors

• On the scalar product

Decomposition of a vector (example in the 2-d space):



$\mathbf{u} \ = \ \mathbf{u}_1 + \mathbf{u}_2$

with $\mathbf{u}_1 \ = \ u_1\, \mathbf{e}_1$ and $\mathbf{u}_2 \ = \ u_2\, \mathbf{e}_2$

$\mathbf{u}_1\, ,\ \mathbf{u}_2$: vector components

$u_1\, ,\ u_2$: coefficients of the vector components

Projection of $\mathbf{u}$ on the directions of $\mathbf{e}_i$:

$$\boxed{u_i \;=\; \mathbf{u} \cdot \mathbf{e}_i}$$

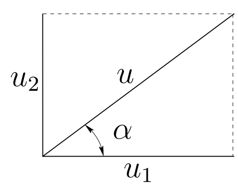
Verification of the projection law:

$$\begin{aligned} \mathbf{u} \cdot \mathbf{e}_i &= (u_k\, \mathbf{e}_k) \cdot \mathbf{e}_i \\ &= u_k\, \delta_{ki} \;=\; u_i \quad \text{q. e. d.} \end{aligned}$$

Calculation of the projections:

$$\begin{aligned} u_1 &= |\mathbf{u}|\, |\mathbf{e}_1| \cos\alpha \\ &= |\mathbf{u}| \cos\alpha \;=\; u \cos\alpha \\ \text{with } u &= |\mathbf{u}| \\ u_2 &= u \cos\beta \\ &= u \cos(90^\circ - \alpha) \;=\; u \sin\alpha \end{aligned}$$

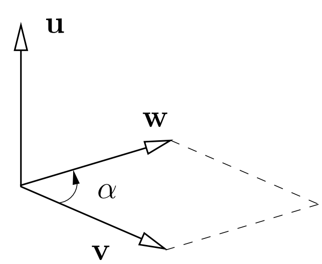
Note: For the values of the vector components, the following relations hold



$$\begin{aligned} u_1 &= u \cos\alpha \\ u_2 &= u \sin\alpha \end{aligned}$$

- **On the vector product**

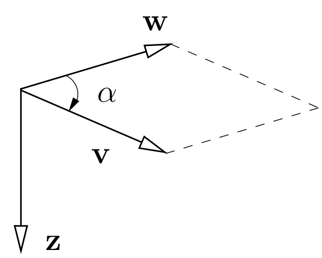
Orientation of the vector $\mathbf{u} \;=\; \mathbf{v} \times \mathbf{w}$:



(1) $\mathbf{u} \perp \{\mathbf{v}\,,\ \mathbf{w}\}$

(2) corkscrew rule (right-hand rule)

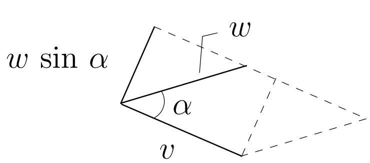
It is obvious that



$$\mathbf{z} \;=\; \mathbf{w} \times \mathbf{v}$$

$$\longrightarrow \quad \mathbf{v} \times \mathbf{w} \;=\; -\,\mathbf{w} \times \mathbf{v}$$

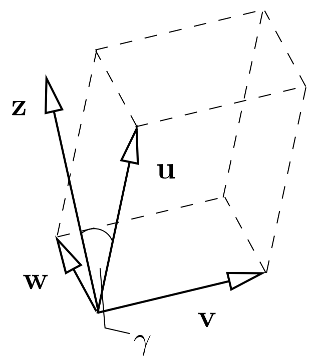
Value of the vector product:



$$\begin{aligned} |\mathbf{v} \times \mathbf{w}| &= |\mathbf{v}|\, |\mathbf{w}| \sin\alpha \\ &= v\,(w \sin\alpha) \end{aligned}$$

Note: The vector $\mathbf{v} \times \mathbf{w}$ is perpendicular to $\mathbf{v}$ and $\mathbf{w}$ (corkscrew orientation). Its value corresponds to the area spanned by $\mathbf{v}$ and $\mathbf{w}$.

Scalar triple product (parallelepidial product):



$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \;=:\; [\,\mathbf{u}\,\mathbf{v}\,\mathbf{w}\,]$$

with $\mathbf{z} = \mathbf{v} \times \mathbf{w}$

follows $\mathbf{u} \cdot \mathbf{z} = |\mathbf{u}|\,|\mathbf{z}|\,\cos\gamma$

$$= z\,(u\,\cos\gamma)$$

with $(u\,\cos\gamma)$: projection of $\mathbf{u}$ on the direction of $\mathbf{z}$

Remark: The parallelepidial product yields the volume of the parallelepiped spanned by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

Remark: The preceding Section on vector calculus and the following Sections on tensor calculus and vector and tensor analysis are mostly written in a basis-free representation. In case that basis systems are taken into consideration, use is made, for simplicity, of the orthonormal basis $\mathbf{e}_i$.

However, to get a deeper inside into the material, the introduction of arbitrary basis systems and, especially, natural basis systems as a subgroup of arbitrary systems is helpful and will therefore be presented in the Appendix to this Treatise.

2 Fundamentals of tensor calculus

Remark: The following statements are related to the proper EUCLIDian vector space $\mathcal{V}^3$ and the corresponding dyadic product space $\mathcal{V}^3 \otimes \mathcal{V}^3 \otimes \cdots \otimes \mathcal{V}^3$ (n times) of n-th order.

2.1 Introduction of the tensor concept

(a) TENSOR CONCEPT AND LINEAR MAPPING

Definition: A 2nd order (2nd rank) tensor $\mathbf{T}$ is a linear mapping which transforms a vector $\mathbf{u}$ uniquely into a vector $\mathbf{w}$:

$$\mathbf{w} = \mathbf{T}\,\mathbf{u}$$

therein:
$$\begin{cases} \mathbf{u},\,\mathbf{w} \;\in \mathcal{V}^3 \quad ; \quad \mathbf{T} \;\in\; \mathcal{L}(\mathcal{V}^3,\,\mathcal{V}^3) \\ \mathcal{L}(\mathcal{V}^3,\,\mathcal{V}^3) \quad : \quad \text{set of all 2nd order tensors or linear mappings of vectors, respectively} \end{cases}$$

Remark: In this treatise on tensor calculus, we follow the notation given by REINT DE BOER in his book "Tensorrechnng für Ingenieure", Springer-Verlag, Berlin 1982.

Reint DE BOER (1935-2010) was a German civil engineer and a mechanics professor at the University of Duisburg-Essen.

(b) TENSOR CONCEPT AND DYADIC PRODUCT SPACE

Definition: There is a "simple tensor" $(\mathbf{a} \otimes \mathbf{b})$ with the property

$$(\mathbf{a} \otimes \mathbf{b})\,\mathbf{c} =: (\mathbf{b} \cdot \mathbf{c})\,\mathbf{a}$$

therein:
$$\begin{cases} \mathbf{a} \otimes \mathbf{b} \;\in\; \mathcal{V}^3 \otimes \mathcal{V}^3 \text{ (dyadic product space)} \\ \quad \otimes \quad : \text{ dyadic product (binary operator of } \mathcal{V}^3 \otimes \mathcal{V}^3) \end{cases}$$

Remark: $(\mathbf{a} \otimes \mathbf{b})$ maps a vector $\mathbf{c}$ onto a vector $\mathbf{d} \;=\; (\mathbf{b} \cdot \mathbf{c})\,\mathbf{a}$ with the direction of $\mathbf{a}$.

Basis notation of a simple tensor:

$$\mathbf{A} := \mathbf{a} \otimes \mathbf{b} = (a_i\,\mathbf{e}_i) \otimes (b_k\,\mathbf{e}_k) = a_i\,b_k\,(\mathbf{e}_i \otimes \mathbf{e}_k)$$

$$\text{with} \quad \begin{cases} a_i\,b_k & : \text{ coefficients of the tensor components} \\ \mathbf{e}_i \otimes \mathbf{e}_k & : \text{ tensor basis} \end{cases}$$

Tensors $\mathbf{A} \in \mathcal{V}^3 \otimes \mathcal{V}^3$ have 9 independent entries (and directions), such as $a_1\,b_3\,(\mathbf{e}_1 \otimes \mathbf{e}_3)$ etc.

Introduction of arbitrary tensors $\mathbf{T} \in \mathcal{V}^3 \otimes \mathcal{V}^3$:

$$\mathbf{T} = \; t_{ik}\,(\mathbf{e}_i \otimes \mathbf{e}_k)$$

$$\text{with} \quad t_{ik} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} \quad : \quad \left\{ \begin{array}{l} \text{matrix of coefficients of } \mathbf{T} \\ \text{with 9 independent entries} \end{array} \right.$$

2.2 Basic rules of tensor algebra

Requirement: $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, ...\} \in \mathcal{V}^3 \otimes \mathcal{V}^3$.

(a) Tensor addition

$$\begin{aligned} \mathbf{A} + \mathbf{B} &= \mathbf{B} + \mathbf{A} && : \text{ commutative law} \\ \mathbf{A} + (\mathbf{B} + \mathbf{C}) &= (\mathbf{A} + \mathbf{B}) + \mathbf{C} && : \text{ associative law} \\ \mathbf{A} + \mathbf{0} &= \mathbf{A} && : \ \mathbf{0} \ : \text{ identical element} \\ \mathbf{A} + (-\mathbf{A}) &= \mathbf{0} && : -\mathbf{A} \ : \text{ inverse element} \end{aligned}$$

Tensor addition with respect to an orthonormal tensor basis:

$$\mathbf{A} = \; a_{ik}\,(\mathbf{e}_i \otimes \mathbf{e}_k), \qquad \mathbf{B} = \; b_{ik}\,(\mathbf{e}_i \otimes \mathbf{e}_k)$$

$$\longrightarrow \; \mathbf{C} = \mathbf{A} + \mathbf{B} = \underbrace{(a_{ik} + b_{ik})}_{c_{ik}}(\mathbf{e}_i \otimes \mathbf{e}_k)$$

Remark: A tensor addition carried out as an addition of the tensor coefficients requires that both tensors have the same tensor basis.

(b) Multiplication of tensors by a scalar

$$\begin{aligned} 1\,\mathbf{A} &= \mathbf{A} && : 1 \ : \text{ identical element} \\ \alpha\,(\beta\,\mathbf{A}) &= (\alpha\,\beta)\,\mathbf{A} && : \text{associative law} \\ (\alpha + \beta)\,\mathbf{A} &= \alpha\,\mathbf{A} + \beta\,\mathbf{A} && : \text{distributive law (with respect to the addition of scalars)} \\ \alpha\,(\mathbf{A} + \mathbf{B}) &= \alpha\,\mathbf{A} + \alpha\,\mathbf{B} && : \text{distributive law (with respect to the addition of tensors)} \\ \alpha\,\mathbf{A} &= \mathbf{A}\,\alpha && : \text{commutative law} \end{aligned}$$

(c) Linear mapping between tensor and vector

The following definitions make use of the linear mapping (cf. 2.1)

$$\mathbf{w} = \mathbf{T}\,\mathbf{u}$$

Remark: In the literature, the linear mapping or the multiplication of a vector by a tensor is also called "contraction".

The following relations hold:

$$\begin{aligned}
\mathbf{A}\,(\mathbf{u}+\mathbf{v}) &= \mathbf{A}\,\mathbf{u}+\mathbf{A}\,\mathbf{v} && : \text{distributive law}\\
\mathbf{A}\,(\alpha\,\mathbf{u}) &= \alpha(\mathbf{A}\,\mathbf{u}) && : \text{associative law}\\
(\mathbf{A}+\mathbf{B})\,\mathbf{u} &= \mathbf{A}\,\mathbf{u}+\mathbf{B}\,\mathbf{u} && : \text{distributive law}\\
(\alpha\,\mathbf{A})\,\mathbf{u} &= \alpha\,(\mathbf{A}\,\mathbf{u}) && : \text{associative law}\\
\mathbf{0}\,\mathbf{u} &= \mathbf{0} && : \ \mathbf{0}\ : \text{zero element of the linear mapping}\\
\mathbf{I}\,\mathbf{u} &= \mathbf{u} && : \ \mathbf{I}\ : \text{identity element of the linear mapping}
\end{aligned}$$

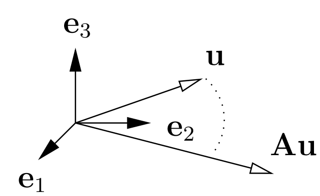
Linear mapping in basis notation:

$$\begin{aligned}
\mathbf{A} &= a_{ik}\,(\mathbf{e}_i\otimes\mathbf{e}_k)\,, \qquad \mathbf{u}=u_i\,\mathbf{e}_i\\
\mathbf{A}\,\mathbf{u} &= (a_{ik}\,\mathbf{e}_i\otimes\mathbf{e}_k)\,(u_j\mathbf{e}_j) \ = \ a_{ik}\,u_j\,(\mathbf{e}_i\otimes\mathbf{e}_k)\,\mathbf{e}_j
\end{aligned}$$

One obtains

$$\mathbf{w} = \mathbf{A}\,\mathbf{u} = a_{ik}\,u_j\,\delta_{kj}\,\mathbf{e}_i = \underbrace{a_{ik}\,u_k}_{w_i}\,\mathbf{e}_i \quad \text{mit} \quad \begin{cases} i & : \ \text{free index (basis index)}\\ k & : \ \text{silent index (double index of } w_i) \end{cases}$$

Remark: In general, a linear mapping $\mathbf{A}$ applied to a vector $\mathbf{u}$ causes both a rotation **and** a stretch of $\mathbf{u}$.



Identity tensor $\mathbf{I}\in\mathcal{V}^3\otimes\mathcal{V}^3$:

$$\mathbf{I} = \delta_{ik}\,\mathbf{e}_i\otimes\mathbf{e}_k = \mathbf{e}_i\otimes\mathbf{e}_i$$

Proof of the defining property:

$$\mathbf{u} = \mathbf{I}\,\mathbf{u} = (\mathbf{e}_i\otimes\mathbf{e}_i)\,u_j\,\mathbf{e}_j = u_j\,(\mathbf{e}_i\otimes\mathbf{e}_i)\,\mathbf{e}_j = u_j\,\delta_{ij}\,\mathbf{e}_i = u_i\,\mathbf{e}_i \quad \text{q. e. d.}$$

Remark: Tensors built from basis vectors are called fundamental tensors. Thus,

> $\mathbf{I}\in\mathcal{V}^3\otimes\mathcal{V}^3$ is the fundamental tensor of 2nd order.

(d) SCALAR PRODUCT OF TENSORS (inner product)

The following relations hold:

$$\begin{aligned}
\mathbf{A}\cdot\mathbf{B} &= \mathbf{B}\cdot\mathbf{A} && : \text{commutative law}\\
\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) &= \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C} && : \text{distributive law}\\
(\alpha\,\mathbf{A})\cdot\mathbf{B} &= \mathbf{A}\cdot(\alpha\,\mathbf{B}) = \alpha\,(\mathbf{A}\cdot\mathbf{B}) && : \text{associative law}\\
\mathbf{A}\cdot\mathbf{B} &= 0 \quad \forall\,\mathbf{A}\,,\ \text{if } \mathbf{B}\equiv\mathbf{0}\\
&\longrightarrow\ \mathbf{A}\cdot\mathbf{A} > \ 0 \text{ for } \mathbf{A}\neq\mathbf{0}
\end{aligned}$$

Scalar product of $\mathbf{A}$ with a simple tensor $\mathbf{a} \otimes \mathbf{b} \, \in \, \mathcal{V}^3 \otimes \mathcal{V}^3$:

$$\mathbf{A} \cdot (\mathbf{a} \otimes \mathbf{b}) = \mathbf{a} \cdot \mathbf{A}\,\mathbf{b}$$

Scalar product of A and B in basis notation:

$$\begin{aligned} \mathbf{A} \; &= \; a_{ik}\,(\mathbf{e}_i \otimes \mathbf{e}_k), \; \mathbf{B} = \; b_{ik}\,(\mathbf{e}_i \otimes \mathbf{e}_k) \\ \alpha \; &= \; \mathbf{A} \cdot \mathbf{B} = \; a_{ik}\,(\mathbf{e}_i \otimes \mathbf{e}_k) \cdot b_{st}(\mathbf{e}_s \otimes \mathbf{e}_t) = \; a_{ik}\,b_{st}(\mathbf{e}_i \otimes \mathbf{e}_k) \cdot (\mathbf{e}_s \otimes \mathbf{e}_t) \end{aligned}$$

One obtains

$$\alpha \;\; = \;\; a_{ik}\,b_{st}\;\delta_{is}\,\delta_{kt} = \; a_{ik}\,b_{ik}$$

Remark: The result of the scalar product is a scalar.

(e) Tensor product of tensors

Definition: The tensor product of tensors yields

$$(\mathbf{A}\mathbf{B})\,\mathbf{v} = \;\mathbf{A}\,(\mathbf{B}\,\mathbf{v})$$

Remark: With this definition, the tensor product of tensors is directly linked to the linear mapping (cf. 2.1 (a)).

The following relations hold:

$$\begin{aligned} (\mathbf{A}\,\mathbf{B})\,\mathbf{C} \;&=\; \mathbf{A}\,(\mathbf{B}\,\mathbf{C}) && : \text{ associative law} \\ \mathbf{A}\,(\mathbf{B} + \mathbf{C}) \;&=\; \mathbf{A}\,\mathbf{B} + \mathbf{A}\,\mathbf{C} && : \text{ distributive law} \\ (\mathbf{A} + \mathbf{B})\,\mathbf{C} \;&=\; \mathbf{A}\,\mathbf{C} + \mathbf{B}\,\mathbf{C} && : \text{ distributive law} \\ \alpha\,(\mathbf{A}\,\mathbf{B}) \;&=\; (\alpha\,\mathbf{A})\,\mathbf{B} \;=\; \mathbf{A}\,(\alpha\,\mathbf{B}) && : \text{ associative law} \\ \mathbf{I}\,\mathbf{T} \;&=\; \mathbf{T}\,\mathbf{I} \;=\; \mathbf{T} && : \; \mathbf{I} \; : \text{ identity element} \\ \mathbf{0}\,\mathbf{T} \;&=\; \mathbf{T}\,\mathbf{0} \;=\; \mathbf{0} && : \; \mathbf{0} \; : \text{ zero element} \end{aligned}$$

Remark: In general, the commutative law is not valid meaning that $\mathbf{A}\,\mathbf{B} \neq \mathbf{B}\,\mathbf{A}$.

Tensor product of simple tensors:

$$\mathbf{A} = \mathbf{a} \otimes \mathbf{b}\,, \quad \mathbf{B} = \mathbf{c} \otimes \mathbf{d}$$

It follows with the above definition

$$\begin{aligned} (\mathbf{A}\,\mathbf{B})\,\mathbf{v} \;&=\; \mathbf{A}\,(\mathbf{B}\,\mathbf{v}) \\ \longrightarrow [\,(\mathbf{a} \otimes \mathbf{b})\,(\mathbf{c} \otimes \mathbf{d})\,]\,\mathbf{v} \;&=\; (\mathbf{a} \otimes \mathbf{b})\,[\,(\mathbf{c} \otimes \mathbf{d})\,\mathbf{v}\,] \\ &=\; (\mathbf{a} \otimes \mathbf{b})\,(\mathbf{d} \cdot \mathbf{v})\,\mathbf{c} \\ &=\; (\mathbf{b} \cdot \mathbf{c})\,(\mathbf{d} \cdot \mathbf{v})\,\mathbf{a} \\ &=\; [\,(\mathbf{b} \cdot \mathbf{c})\,(\mathbf{a} \otimes \mathbf{d})\,]\,\mathbf{v} \end{aligned}$$

Consequence:

$$(\mathbf{a} \otimes \mathbf{b})\,(\mathbf{c} \otimes \mathbf{d}) \;=\; (\mathbf{b} \cdot \mathbf{c})\,\mathbf{a} \otimes \mathbf{d}$$

Tensor product in basis notation:

$$\begin{aligned}
\mathbf{A}\,\mathbf{B} &= a_{ik}\,(\mathbf{e}_i \otimes \mathbf{e}_k)\,b_{st}(\mathbf{e}_s \otimes \mathbf{e}_t) \\
&= a_{ik}\,b_{st}\,(\mathbf{e}_i \otimes \mathbf{e}_k)\,(\mathbf{e}_s \otimes \mathbf{e}_t) \\
&= a_{ik}\,b_{st}\;\delta_{ks}(\mathbf{e}_i \otimes \mathbf{e}_t) \\
&= a_{ik}\,b_{kt}\;(\mathbf{e}_i \otimes \mathbf{e}_t)
\end{aligned}$$

Remark: The result of a tensor product is a tensor.

2.3 Specific tensors and operations

(a) TRANSPOSED TENSOR

Definition: The transposed tensor $\mathbf{A}^T$ belonging to $\mathbf{A}$ exhibits the property

$$\mathbf{w} \cdot (\mathbf{A}\,\mathbf{u}) \;= (\mathbf{A}^T\,\mathbf{w}) \cdot \mathbf{u}$$

The following relations hold:

$$\begin{aligned}
(\mathbf{A} + \mathbf{B})^T &= \mathbf{A}^T + \mathbf{B}^T \\
(\alpha\,\mathbf{A})^T &= \alpha\,\mathbf{A}^T \\
(\mathbf{A}\,\mathbf{B})^T &= \mathbf{B}^T\,\mathbf{A}^T
\end{aligned}$$

Transposition of a simple tensor $\mathbf{a} \otimes \mathbf{b}$:

It follows with the above definition

$$\begin{aligned}
\mathbf{w} \cdot\;(\mathbf{a} \otimes \mathbf{b})\,\mathbf{u} &= \mathbf{w} \cdot (\mathbf{b} \cdot \mathbf{u})\,\mathbf{a} \\
&= (\mathbf{w} \cdot \mathbf{a})\,(\mathbf{b} \cdot \mathbf{u}) \\
&= (\mathbf{b} \otimes \mathbf{a})\,\mathbf{w} \cdot \mathbf{u} \\
\longrightarrow (\mathbf{a} \otimes \mathbf{b})^T &= \mathbf{b} \otimes \mathbf{a}
\end{aligned}$$

Transposed tensor in basis notation:

$$\begin{aligned}
\mathbf{A} &= a_{ik}\,(\mathbf{e}_i \otimes \mathbf{e}_k) \\
\longrightarrow\;\mathbf{A}^T &= a_{ik}\,(\mathbf{e}_k \otimes \mathbf{e}_i) \quad &&: \text{exchanging the basis vectors} \\
&= a_{ki}\,(\mathbf{e}_i \otimes \mathbf{e}_k) \quad &&: \text{exchanging the indices of the tensor coefficients}
\end{aligned}$$

Note: The transposition of a tensor $\mathbf{A} \in \mathcal{V}^3 \otimes \mathcal{V}^3$ can be carried out by an exchange of the tensor basis **or** by an exchange of the subscripts of the tensor coefficients.

(b) Symmetric and skew-symmetric tensors

Definition: A tensor $\mathbf{A} \in \mathcal{V}^3 \otimes \mathcal{V}^3$ is symmetric, if

$$\mathbf{A} = \mathbf{A}^T$$

and skew-symmetric (antimetric), if

$$\mathbf{A} = -\mathbf{A}^T$$

Symmetric and skew-symmetric parts of an arbitrary tensor $\mathbf{A} \in \mathcal{V}^3 \otimes \mathcal{V}^3$:

$$\begin{aligned} \operatorname{sym}\mathbf{A} &= \tfrac{1}{2}(\mathbf{A} + \mathbf{A}^T) \\ \operatorname{skw}\mathbf{A} &= \tfrac{1}{2}(\mathbf{A} - \mathbf{A}^T) \\ \longrightarrow \mathbf{A} &= \operatorname{sym}\mathbf{A} + \operatorname{skw}\mathbf{A} \end{aligned}$$

Properties of symmetric and skew-symmetric tensors:

$$\begin{aligned} \mathbf{w} \cdot (\operatorname{sym}\mathbf{A})\,\mathbf{v} &= (\operatorname{sym}\mathbf{A})\,\mathbf{w} \cdot \mathbf{v} \\ \mathbf{v} \cdot (\operatorname{skw}\mathbf{A})\,\mathbf{v} &= -(\operatorname{skw}\mathbf{A})\,\mathbf{v} \cdot \mathbf{v} = 0 \end{aligned}$$

Symmetric tensors with the property of positive definiteness:

- $\operatorname{sym}\mathbf{A}$ is positive definite, if $\quad \operatorname{sym}\mathbf{A} \cdot (\mathbf{v} \otimes \mathbf{v}) = \mathbf{v} \cdot (\operatorname{sym}\mathbf{A})\,\mathbf{v} > 0$
- $\operatorname{sym}\mathbf{A}$ is positive semi-definite, if $\quad \operatorname{sym}\mathbf{A} \cdot (\mathbf{v} \otimes \mathbf{v}) = \mathbf{v} \cdot (\operatorname{sym}\mathbf{A})\,\mathbf{v} \geq 0$

(c) Inverse tensor

Definition: If $\mathbf{A}^{-1}$ inverse to $\mathbf{A}$ exists, it exhibits the property

$$\mathbf{v} = \mathbf{A}\,\mathbf{w} \quad \longleftrightarrow \quad \mathbf{w} = \mathbf{A}^{-1}\,\mathbf{v}$$

The following relations hold:

$$\begin{aligned} \mathbf{A}\,\mathbf{A}^{-1} &= \mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I} \\ (\mathbf{A}^{-1})^T &= (\mathbf{A}^T)^{-1} =: \mathbf{A}^{T-1} (= \mathbf{A}^{-T}) \\ (\mathbf{A}\,\mathbf{B})^{-1} &= \mathbf{B}^{-1}\,\mathbf{A}^{-1} \end{aligned}$$

Remark: The computation of the inverse tensor in basis notation is carried out by use of the "double cross product" (outer tensor product of tensors), cf. Subsection 2.8.

(d) Orthogonal tensor

Definition: An orthogonal tensor $\mathbf{Q} \in \mathcal{V}^3 \otimes \mathcal{V}^3$ exhibits the property

$$\mathbf{Q}^{-1} = \mathbf{Q}^T \quad \longleftrightarrow \quad \mathbf{Q}\,\mathbf{Q}^T = \mathbf{I}$$

$$\text{Additionally} \begin{cases} (\det \mathbf{Q})^2 = 1 & : \text{orthogonality} \\ \det \mathbf{Q} = 1 & : \text{proper orthogonality} \end{cases}$$

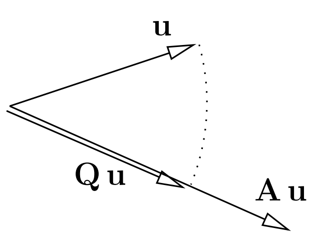
Remark: The computation of the determinant of 2nd order tensors is defined with the aid of the double cross product, cf. 2.8.

Properties of orthogonal tensors:

$$\begin{aligned} \mathbf{Q}\,\mathbf{v} \cdot \mathbf{Q}\,\mathbf{w} &= \mathbf{Q}^T \mathbf{Q}\,\mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w} \\ \longrightarrow \; \mathbf{Q}\,\mathbf{u} \cdot \mathbf{Q}\,\mathbf{u} &= \mathbf{u} \cdot \mathbf{u} \end{aligned}$$

Remark: Linear mapping with $\mathbf{Q}$ preserves the norm of the respective vector.

Illustration:



generally, a linear mapping with $\mathbf{A} \in \mathcal{V}^3 \otimes \mathcal{V}^3$ causes a rotation **and** a stretch

especially, a linear mapping with $\mathbf{Q} \in \mathcal{V}^3 \otimes \mathcal{V}^3$ causes **only** a rotation

(e) Trace of a tensor

Definition: The trace $\text{tr}\,\mathbf{A}$ of a tensor $\mathbf{A} \in \mathcal{V}^3 \otimes \mathcal{V}^3$ is the scalar product

$$\text{tr}\,\mathbf{A} = \mathbf{A} \cdot \mathbf{I}$$

The following relations hold:

$$\begin{aligned} \text{tr}\,(\alpha\,\mathbf{A}) &= \alpha\,\text{tr}\,\mathbf{A} \\ \text{tr}\,(\mathbf{a} \otimes \mathbf{b}) &= \mathbf{a} \cdot \mathbf{b} \\ \text{tr}\,\mathbf{A}^T &= \text{tr}\,\mathbf{A} \\ \text{tr}\,(\mathbf{A}\,\mathbf{B}) &= \text{tr}\,(\mathbf{B}\mathbf{A}) \\ \longrightarrow \; (\mathbf{A}\,\mathbf{B}) \cdot \mathbf{I} &= \mathbf{B} \cdot \mathbf{A}^T = \mathbf{B}^T \cdot \mathbf{A} \\ \text{tr}\,(\mathbf{A}\,\mathbf{B}\,\mathbf{C}) &= \text{tr}\,(\mathbf{B}\,\mathbf{C}\,\mathbf{A}) = \text{tr}\,(\mathbf{C}\,\mathbf{A}\,\mathbf{B}) \end{aligned}$$

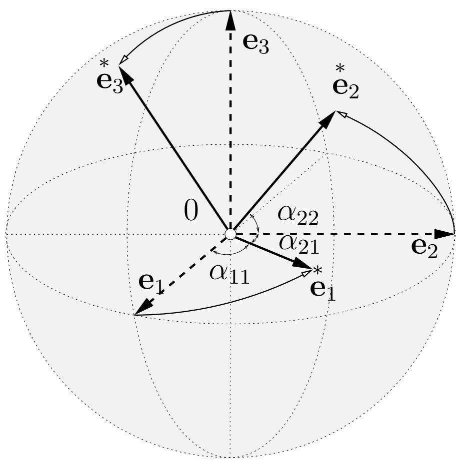
2.4 Change of the basis

Remark: The goal is to find a relation between vectors and tensors which belong to different basis systems.

here: Restriction to orthonormal basis systems which are rotated against each other.

(A) Rotation of the basis system

Illustration:



$\{0, \mathbf{e}_i\}$: basis system

$\{0, \overset{*}{\mathbf{e}}_i\}$: rotated basis system

$\{\alpha_{ik}\}$: angle between the basis vectors $\mathbf{e}_i$ and $\overset{*}{\mathbf{e}}_k$

Development of the transformation tensor:

The following relations hold:

$$\overset{*}{\mathbf{e}}_i = \mathbf{I}\,\overset{*}{\mathbf{e}}_i \quad \text{and} \quad \mathbf{I} = \mathbf{e}_j \otimes \mathbf{e}_j$$

Thus,

$$\overset{*}{\mathbf{e}}_i = (\mathbf{e}_j \otimes \mathbf{e}_j)\,\overset{*}{\mathbf{e}}_i = (\mathbf{e}_j \cdot \overset{*}{\mathbf{e}}_i)\,\mathbf{e}_j$$

using $\overset{*}{\mathbf{e}}_i = \delta_{ik}\,\overset{*}{\mathbf{e}}_k$ with $\delta_{ik} = \mathbf{e}_i \cdot \mathbf{e}_k$ leads to

$$\overset{*}{\mathbf{e}}_i = (\mathbf{e}_j \cdot \delta_{ik}\,\overset{*}{\mathbf{e}}_k)\,\mathbf{e}_j = (\mathbf{e}_j \cdot \overset{*}{\mathbf{e}}_k)\,(\mathbf{e}_i \cdot \mathbf{e}_k)\,\mathbf{e}_j$$

and one obtains

$$\overset{*}{\mathbf{e}}_i = (\mathbf{e}_j \cdot \overset{*}{\mathbf{e}}_k)\,(\mathbf{e}_j \otimes \mathbf{e}_k)\,\mathbf{e}_i =: \mathbf{R}\,\mathbf{e}_i \quad \text{with} \quad \mathbf{R} = (\mathbf{e}_j \cdot \overset{*}{\mathbf{e}}_k)\,\mathbf{e}_j \otimes \mathbf{e}_k$$

Remark: $\mathbf{R}$ is the transformation tensor which transforms the basis vectors $\mathbf{e}_i$ into the basis vectors $\overset{*}{\mathbf{e}}_i$.

Coefficient matrix R_{jk}:

$$R_{jk} = \mathbf{e}_j \cdot \overset{*}{\mathbf{e}}_k = |\mathbf{e}_j|\,|\overset{*}{\mathbf{e}}_k|\,\cos\sphericalangle(\mathbf{e}_j; \overset{*}{\mathbf{e}}_k) = \cos\alpha_{jk} \quad \text{with } |\mathbf{e}_j| = |\overset{*}{\mathbf{e}}_k| = 1$$

Remark: R_{jk} contains the 9 cosines of the angles between the directions of the basis vectors $\mathbf{e}_j$ and $\overset{*}{\mathbf{e}}_k$.

Orthogonality of the transformation tensor:

Remark: By $\mathbf{R}$, the basis vectors $\mathbf{e}_i$ are only rotated towards $\overset{*}{\mathbf{e}}_i$. Thus, $\mathbf{R}$ is an orthogonal tensor.

Orthogonality condition:

$$\begin{aligned} \mathbf{R}\,\mathbf{R}^T \overset{!}{=} \mathbf{I} \quad &= R_{jk}(\mathbf{e}_j \otimes \mathbf{e}_k)\, R_{pn}\,(\mathbf{e}_n \otimes \mathbf{e}_p) = R_{jk}\, R_{pn}\, \delta_{kn}\, \mathbf{e}_j \otimes \mathbf{e}_p \\ &= R_{jk}\, R_{pk}\,(\mathbf{e}_j \otimes \mathbf{e}_p) \end{aligned}$$

It follows with $\mathbf{I} = \delta_{jp}\,(\mathbf{e}_j \otimes \mathbf{e}_p)$ by comparison of coefficients

$$\boxed{R_{jk}\, R_{pk} = \delta_{jp}} \qquad (*)$$

Remark: $(*)$ contains 6 constraints for the 9 cosines ($\mathbf{R}\,\mathbf{R}^T = \operatorname{sym}(\mathbf{R}\,\mathbf{R}^T)$), i. e. only 3 of 9 trigonometrical functions are independent. Thus, the rotation of the basis system is defined by 3 angles.

(B) Introduction of "CARDANO angles"

Idea: Rotation around 3 axes which are given by the basis directions $\mathbf{e}_i$. This procedure goes back to Gerolamo Cardano.

Gerolamo Cardano (1501-1576) is considered one of the last great universal scholars of the Renaissance with an astonishing international reputation in various fields, such as medicine, mathematics, philosophy, physics, chemistry, and engineering.

Procedure: The rotation of the basis system is carried out by 3 independent rotations around the axes $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. Each rotation is expressed by a transformation tensor $\mathbf{R}_i$ $(\mathrm{i} = 1, 2, 3)$.

Rotation of $\mathbf{e}_i$ around $\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1$:

$$\overset{*}{\mathbf{e}}_i = \{\mathbf{R}_1\,[\mathbf{R}_2\,(\mathbf{R}_3\,\mathbf{e}_i)]\} = \overset{*}{\mathbf{R}}\,\mathbf{e}_i \qquad \text{with} \quad \overset{*}{\mathbf{R}} = \mathbf{R}_1\,\mathbf{R}_2\,\mathbf{R}_3$$

Rotation of $\mathbf{e}_i$ around $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$:

$$\bar{\mathbf{e}}_i = \{\mathbf{R}_3\,[\mathbf{R}_2\,(\mathbf{R}_1\,\mathbf{e}_i)]\} = \bar{\mathbf{R}}\,\mathbf{e}_i \qquad \text{with} \quad \bar{\mathbf{R}} = \mathbf{R}_3\,\mathbf{R}_2\,\mathbf{R}_1$$

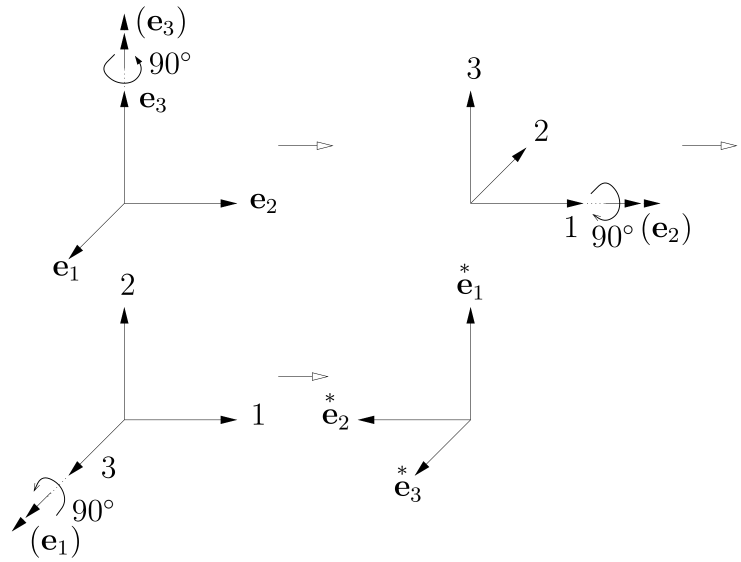
Obviously,

$$\overset{*}{\mathbf{R}} \neq \bar{\mathbf{R}} \quad \longrightarrow \quad \overset{*}{\mathbf{e}} \neq \bar{\mathbf{e}}_i$$

Remark: The result of the orthogonal transformation depends on the sequence of the rotations.

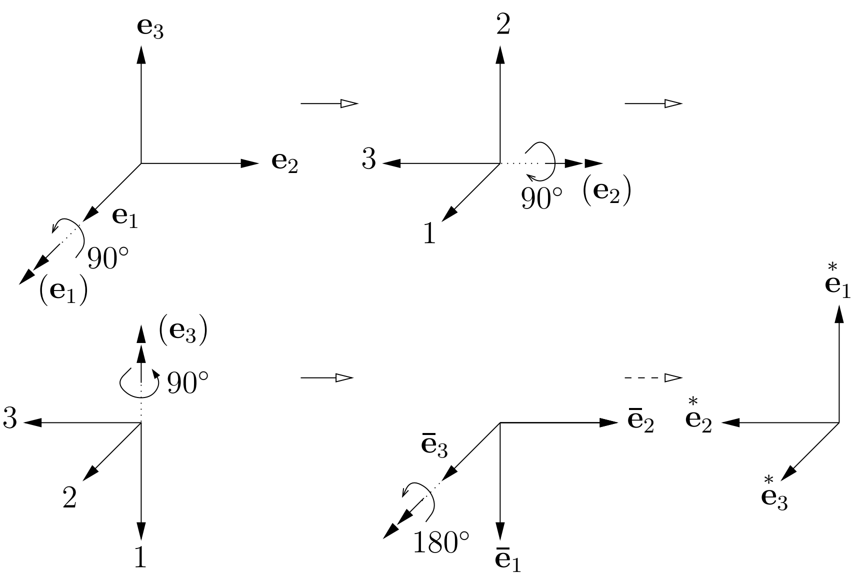
Illustration:

(a) Rotation around $\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1$ (e. g. each by 90°)



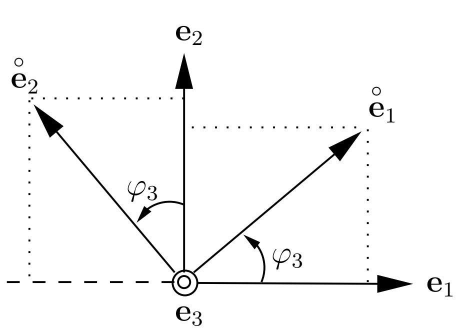
(b) Rotation around $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ (e. g. each by 90°)

with



Definition of the orthogonal rotation tensors $\mathbf{R}_i$

(a) Rotation around the $\mathbf{e}_3$-axis



The following relations hold:

$$\overset{\circ}{\mathbf{e}}_1 = \cos\varphi_3\,\mathbf{e}_1 + \sin\varphi_3\,\mathbf{e}_2$$

$$\overset{\circ}{\mathbf{e}}_2 = -\sin\varphi_3\,\mathbf{e}_1 + \cos\varphi_3\,\mathbf{e}_2$$

$$\overset{\circ}{\mathbf{e}}_3 = \mathbf{e}_3$$

In general,

$$\overset{\circ}{\mathbf{e}}_i = \mathbf{R}_3\,\mathbf{e}_i = R_{3jk}\,(\mathbf{e}_j \otimes \mathbf{e}_k)\,\mathbf{e}_i = R_{3jk}\,\delta_{ki}\,\mathbf{e}_j = R_{3ji}\,\mathbf{e}_j$$

Thus, by comparison of coefficients

$$\mathbf{R}_3 = R_{3ji}\,(\mathbf{e}_j \otimes \mathbf{e}_i) \quad \text{with} \quad R_{3ji} = \begin{bmatrix} \cos\varphi_3 & -\sin\varphi_3 & 0 \\ \sin\varphi_3 & \cos\varphi_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Rotation around the $\mathbf{e}_2$- and $\mathbf{e}_1$-axis

Analogously,

$$\mathbf{R}_2 = R_{2ji}\,(\mathbf{e}_j \otimes \mathbf{e}_i) \quad \text{with} \quad R_{2ji} = \begin{bmatrix} \cos\varphi_2 & 0 & \sin\varphi_2 \\ 0 & 1 & 0 \\ -\sin\varphi_2 & 0 & \cos\varphi_2 \end{bmatrix}$$

$$\mathbf{R}_1 = R_{1ji}\,(\mathbf{e}_j \otimes \mathbf{e}_i) \quad \text{with} \quad R_{1ji} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_1 & -\sin\varphi_1 \\ 0 & \sin\varphi_1 & \cos\varphi_1 \end{bmatrix}$$

Remark: The rotation tensor $\mathbf{R}$ can be composed of single rotations under consideration of the rotation sequence.

(c) Definition of the total rotation $\mathbf{R}$

(c_1) it follows from the rotation of $\mathbf{e}_i$ around $\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1$ that

$$\begin{aligned} \mathbf{R} \longrightarrow \overset{*}{\mathbf{R}} &= \mathbf{R}_1\,\mathbf{R}_2\,\mathbf{R}_3 \\ &= R_{1ij}\,(\mathbf{e}_i \otimes \mathbf{e}_j)\,R_{2no}\,(\mathbf{e}_n \otimes \mathbf{e}_o)\,R_{3pq}\,(\mathbf{e}_p \otimes \mathbf{e}_q) \\ &= R_{1ij}\,R_{2no}\,R_{3pq}\,\delta_{jn}\,\delta_{op}\,(\mathbf{e}_i \otimes \mathbf{e}_q) \\ &= \underbrace{R_{1ij}\,R_{2jo}\,R_{3oq}}_{\overset{*}{R}_{iq}}\,(\mathbf{e}_i \otimes \mathbf{e}_q) \end{aligned}$$

with

$$\overset{*}{R}_{iq} = \begin{bmatrix} \cos\varphi_2\,\cos\varphi_3 & -\cos\varphi_2\,\sin\varphi_3 & \sin\varphi_2 \\ \sin\varphi_1\,\sin\varphi_2\,\cos\varphi_3 + \cos\varphi_1\,\sin\varphi_3 & -\sin\varphi_1\,\sin\varphi_2\,\sin\varphi_3 + \cos\varphi_1\,\cos\varphi_3 & -\sin\varphi_1\,\cos\varphi_2 \\ -\cos\varphi_1\,\sin\varphi_2\,\cos\varphi_3 + \sin\varphi_1\,\sin\varphi_3 & \cos\varphi_1\,\sin\varphi_2\,\sin\varphi_3 + \sin\varphi_1\,\cos\varphi_3 & \cos\varphi_1\,\cos\varphi_2 \end{bmatrix}$$

(c_2) it follows from the rotation of $\mathbf{e}_i$ around $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ that

$$\begin{aligned} \mathbf{R} \longrightarrow \bar{\mathbf{R}} &= \mathbf{R}_3\,\mathbf{R}_2\,\mathbf{R}_1 \\ &= \underbrace{R_{3ij}\,R_{2jo}\,R_{1oq}}_{\bar{R}_{iq}}\,(\mathbf{e}_i \otimes \mathbf{e}_q) \end{aligned}$$

with

$$\bar{R}_{iq} = \begin{bmatrix} \cos\varphi_2\,\cos\varphi_3 & \sin\varphi_1\,\sin\varphi_2\,\cos\varphi_3 - \cos\varphi_1\,\sin\varphi_3 & \cos\varphi_1\,\sin\varphi_2\,\cos\varphi_3 + \sin\varphi_1\,\sin\varphi_3 \\ \cos\varphi_2\,\sin\varphi_3 & \sin\varphi_1\,\sin\varphi_2\,\sin\varphi_3 + \cos\varphi_1\,\cos\varphi_3 & \cos\varphi_1\,\sin\varphi_2\,\sin\varphi_3 - \sin\varphi_1\,\cos\varphi_3 \\ -\sin\varphi_2 & \sin\varphi_1\,\cos\varphi_2 & \cos\varphi_1\,\cos\varphi_2 \end{bmatrix}$$

Orthogonality of "CARDANO rotation tensors":

For all $\mathbf{R} \in \{\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3, \overset{*}{\mathbf{R}}, \bar{\mathbf{R}}\}$, the following relations hold

$$\mathbf{R}^{-1} = \mathbf{R}^T, \text{ i. e. } \mathbf{R}\,\mathbf{R}^T = \mathbf{I} \quad \text{and} \quad (\det\mathbf{R})^2 = 1 \quad \longrightarrow \quad \text{orthogonality}$$

Furthermore, all rotation tensors hold the following relation

$$\det\mathbf{R} = 1 \quad : \text{ "proper" orthogonality}$$

Remark: A basis transformation with "non-proper" orthogonal transformations ($\det\mathbf{R} = -1$) transforms a "right-handed" into a "left-handed" basis system.

Example:

here: Investigation of the orthogonality properties of $\mathbf{R}_3 = \mathbf{R}_{3ij}\,(\mathbf{e}_i \otimes \mathbf{e}_j)$

$$\text{with} \quad R_{3ij} = \begin{bmatrix} \cos\varphi_3 & -\sin\varphi_3 & 0 \\ \sin\varphi_3 & \cos\varphi_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

One looks at

$$\begin{aligned} \mathbf{R}_3\,\mathbf{R}_3^T &= R_{3ij}(\mathbf{e}_i \otimes \mathbf{e}_j)\,R_{3on}(\mathbf{e}_n \otimes \mathbf{e}_o) \\ &= R_{3ij}\,R_{3on}\,\delta_{jn}\,(\mathbf{e}_i \otimes \mathbf{e}_o) = R_{3in}\,R_{3on}\,(\mathbf{e}_i \otimes \mathbf{e}_o) \end{aligned}$$

where

$$R_{3in}\,R_{3on} = \begin{bmatrix} \sin^2\varphi_3 + \cos^2\varphi_3 & 0 & 0 \\ 0 & \sin^2\varphi_3 + \cos^2\varphi_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \delta_{io}$$

and one obtains

$$\mathbf{R}_3\,\mathbf{R}_3^T = \delta_{io}\,(\mathbf{e}_i \otimes \mathbf{e}_o) = \mathbf{I} \quad \text{q. e. d.}$$

Furthermore,

$$\det\mathbf{R}_3 := \det(R_{3ij}) = 1 \quad \longrightarrow \quad \mathbf{R}_3 \text{ is proper orthogonal}$$

Description of rotation tensors:

In general, the transformation between basis systems $\bar{\mathbf{e}}_i$ and basis systems $\overset{\circ}{\mathbf{e}}_i$ satisfies the following relation:

$$\overset{\circ}{\mathbf{e}}_i = \bar{\mathbf{R}}\,\bar{\mathbf{e}}_i \quad \text{with} \quad \bar{\mathbf{R}} = \bar{R}_{ik}\,\bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_k$$

$$\longrightarrow \quad \bar{\mathbf{e}}_i = \bar{\mathbf{R}}^T\,\overset{\circ}{\mathbf{e}}_i \quad \text{with} \quad \bar{\mathbf{R}}^{-1} \equiv \bar{\mathbf{R}}^T$$

Otherwise,

$$\bar{\mathbf{e}}_i = \mathring{\mathbf{R}}\,\mathring{\mathbf{e}}_i \quad \text{with} \quad \mathring{\mathbf{R}} = \mathring{R}_{ik}\,\mathring{\mathbf{e}}_i \otimes \mathring{\mathbf{e}}_k$$

Consequence: By comparing both relations, it follows that

$$\mathring{\mathbf{R}} = \bar{\mathbf{R}}^T, \;\; \text{i. e.,} \;\; \mathring{R}_{ik}\,\mathring{\mathbf{e}}_i \otimes \mathring{\mathbf{e}}_k = (\bar{R}_{ik})^T\,\bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_k \quad \longrightarrow \quad \mathring{R}_{ik} = \bar{R}_{ki}$$

In particular,

$$\begin{aligned} \mathring{\mathbf{R}} &= \mathring{R}_{ik}\,(\mathring{\mathbf{e}}_i \otimes \mathring{\mathbf{e}}_k) = \mathring{R}_{ik}\,(\bar{\mathbf{R}}\,\bar{\mathbf{e}}_i \otimes \bar{\mathbf{R}}\,\bar{\mathbf{e}}_k) \\ &= \mathring{R}_{ik}\,\bar{R}_{ni}\,\bar{\mathbf{e}}_n \otimes \bar{R}_{pk}\,\bar{\mathbf{e}}_p = (\bar{R}_{ni}\,\mathring{R}_{ik}\,\bar{R}_{pk})\,\bar{\mathbf{e}}_n \otimes \bar{\mathbf{e}}_p \overset{!}{=} \bar{R}_{pn}\,\bar{\mathbf{e}}_n \otimes \bar{\mathbf{e}}_p = \bar{\mathbf{R}}^T \end{aligned}$$

$$\longrightarrow \quad \boxed{\bar{R}_{ni}\,\mathring{R}_{ik}\,\bar{R}_{pk} \overset{!}{=} \bar{R}_{pn} \quad \longleftrightarrow \quad \bar{R}_{ni}\,\mathring{R}_{ik} = \delta_{nk}}$$

Remark: The coefficient matrices $\bar{R}_{ni}$ and $\mathring{R}_{ik}$ are inverse to each other, i. e., in general, $\bar{R}_{ni}\,\mathring{R}_{ik} = \delta_{nk}$ implies 6 equations for the 9 unknown coefficients $\mathring{R}_{ik}$. Due to $\bar{\mathbf{R}}^{-1} = \bar{\mathbf{R}}^T$, one has $\bar{R}_{ni}^{-1} = (\bar{R}_{ni})^T = \bar{R}_{in}$, i. e.

$$\boxed{\mathring{R}_{ik} = (\bar{R}_{ik})^T = \bar{R}_{ki}}$$

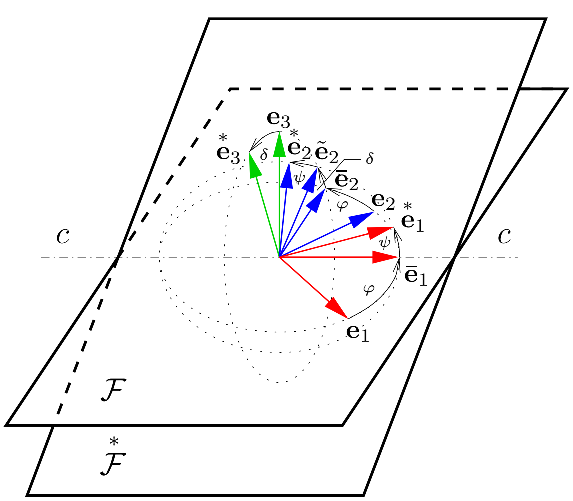
(C) Introduction of EULER angles

Leonhard Euler (1707-1783) was a Swiss mathematician, physicist, astronomer, geographer, logician and engineer.

Remark: Rotation of a basis system $\mathbf{e}_i$ around three specific axes.

Introduction of 3 specific angles around $\mathbf{e}_3$, $\bar{\mathbf{e}}_1$, $\tilde{\mathbf{e}}_3 = \overset{*}{\mathbf{e}}_3$

Illustration:

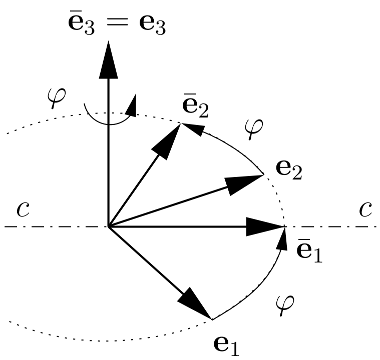


Idea: Given are 2 planes $\mathcal{F}$ and $\overset{*}{\mathcal{F}}$ with in-plane vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\overset{*}{\mathbf{e}}_1$, $\overset{*}{\mathbf{e}}_2$ and surface normals $\mathbf{e}_3$ and $\overset{*}{\mathbf{e}}_3$. The basis systems $\mathbf{e}_i$ and $\overset{*}{\mathbf{e}}_i$ are related to each other by the Eulerian rotation tensor $\mathbf{R}$:

$$\overset{*}{\mathbf{e}}_i := \mathbf{R}\,\mathbf{e}_i$$

1st step:

Rotation of $\mathbf{e}_i$ in plane $\mathcal{F}$ around $\mathbf{e}_3$ with the angle φ, such that $\bar{\mathbf{e}}_i$ is directed along the line $c-c$. This yields the rotation tensor



$$\mathbf{R}_3 = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{e}_j \otimes \mathbf{e}_k \,.$$

Then, the new system $\bar{\mathbf{e}}_i$ is computed as follows

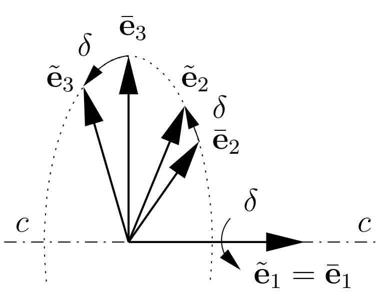
$$\bar{\mathbf{e}}_i = \mathbf{R}_3\,\mathbf{e}_i = R_{3jk}\,(\mathbf{e}_j \otimes \mathbf{e}_k)\,\mathbf{e}_i = R_{3ji}\,\mathbf{e}_j \,.$$

Thus,

$$\begin{aligned} \bar{\mathbf{e}}_1 &= R_{3j1}\,\mathbf{e}_j = \cos\varphi\,\mathbf{e}_1 + \sin\varphi\,\mathbf{e}_2 \\ \bar{\mathbf{e}}_2 &= R_{3j2}\,\mathbf{e}_j = -\sin\varphi\,\mathbf{e}_1 + \cos\varphi\,\mathbf{e}_2 \\ \bar{\mathbf{e}}_3 &= R_{3j3}\,\mathbf{e}_j = \mathbf{e}_3 \,. \end{aligned}$$

2nd step:

Rotation of $\bar{\mathbf{e}}_i$ around $\bar{\mathbf{e}}_1$ with the angle δ, such that $\tilde{\mathbf{e}}_2$ lies in the plane $\overset{*}{\mathcal{F}}$, and $\tilde{\mathbf{e}}_3$ is directed normal to the plane $\overset{*}{\mathcal{F}}$. This yields the rotation tensor



$$\bar{\mathbf{R}}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\delta & -\sin\delta \\ 0 & \sin\delta & \cos\delta \end{bmatrix} \bar{\mathbf{e}}_j \otimes \bar{\mathbf{e}}_k \,.$$

Then, the new system $\tilde{\mathbf{e}}_i$ is computed as follows

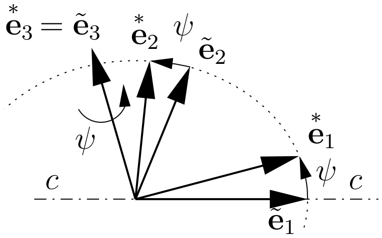
$$\tilde{\mathbf{e}}_i = \bar{\mathbf{R}}_1\,\bar{\mathbf{e}}_i = \bar{R}_{1jk}\,(\bar{\mathbf{e}}_j \otimes \bar{\mathbf{e}}_k)\,\bar{\mathbf{e}}_i = \bar{R}_{1ji}\,\bar{\mathbf{e}}_j \,.$$

Thus,

$$\begin{aligned} \tilde{\mathbf{e}}_1 &= \bar{R}_{1j1}\,\bar{\mathbf{e}}_j = \bar{\mathbf{e}}_1 \\ \tilde{\mathbf{e}}_2 &= \bar{R}_{1j2}\,\bar{\mathbf{e}}_j = \cos\delta\,\bar{\mathbf{e}}_2 + \sin\delta\,\bar{\mathbf{e}}_3 \\ \tilde{\mathbf{e}}_3 &= \bar{R}_{1j3}\,\bar{\mathbf{e}}_j = -\sin\delta\,\bar{\mathbf{e}}_2 + \cos\delta\,\bar{\mathbf{e}}_3 \,. \end{aligned}$$

3rd step:

Rotation of $\tilde{\mathbf{e}}_i$ in plane $\overset{*}{\mathcal{F}}$ around $\tilde{\mathbf{e}}_3$ with the angle ψ. This yields the rotation tensor



$$\tilde{\mathbf{R}}_3 = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tilde{\mathbf{e}}_j \otimes \tilde{\mathbf{e}}_k \,.$$

Then, the new system $\overset{*}{\mathbf{e}}_i$ is computed as follows

$$\overset{*}{\mathbf{e}}_i = \tilde{\mathbf{R}}_3\,\tilde{\mathbf{e}}_i = \tilde{R}_{3jk}\,(\tilde{\mathbf{e}}_j \otimes \tilde{\mathbf{e}}_k)\,\tilde{\mathbf{e}}_i = \tilde{R}_{3ji}\,\tilde{\mathbf{e}}_j \,.$$

Thus,

$$\begin{aligned}
\overset{*}{\mathbf{e}}_1 &= \tilde{R}_{3j1}\,\tilde{\mathbf{e}}_j = \cos\psi\,\tilde{\mathbf{e}}_1 + \sin\psi\,\tilde{\mathbf{e}}_2 \\
\overset{*}{\mathbf{e}}_2 &= \tilde{R}_{3j2}\,\tilde{\mathbf{e}}_j = -\sin\psi\,\tilde{\mathbf{e}}_1 + \cos\psi\,\tilde{\mathbf{e}}_2 \\
\overset{*}{\mathbf{e}}_3 &= \tilde{R}_{3j3}\,\tilde{\mathbf{e}}_j = \tilde{\mathbf{e}}_3\,.
\end{aligned}$$

Summary:

(a) Inserting $\tilde{\mathbf{e}}_i = \bar{\mathbf{R}}_1\,\bar{\mathbf{e}}_i$

$$\begin{aligned}
\overset{*}{\mathbf{e}}_1 &= \cos\psi\,\bar{\mathbf{e}}_1 + \sin\psi\,(\cos\delta\,\bar{\mathbf{e}}_2 + \sin\delta\,\bar{\mathbf{e}}_3) \\
\overset{*}{\mathbf{e}}_2 &= -\sin\psi\,\bar{\mathbf{e}}_1 + \cos\psi\,(\cos\delta\,\bar{\mathbf{e}}_2 + \sin\delta\,\tilde{\mathbf{e}}_3) \\
\overset{*}{\mathbf{e}}_3 &= \tilde{\mathbf{e}}_3 = -\sin\delta\,\bar{\mathbf{e}}_2 + \cos\delta\,\bar{\mathbf{e}}_3
\end{aligned}$$

Result:

$$\begin{aligned}
\overset{*}{\mathbf{e}}_1 &= \cos\psi\,\bar{\mathbf{e}}_1 + \sin\psi\,\cos\delta\,\bar{\mathbf{e}}_2 + \sin\psi\,\sin\delta\,\bar{\mathbf{e}}_3 \\
\overset{*}{\mathbf{e}}_2 &= -\sin\psi\,\bar{\mathbf{e}}_1 + \cos\psi\,\cos\delta\,\bar{\mathbf{e}}_2 + \cos\psi\,\sin\delta\,\tilde{\mathbf{e}}_3 \\
\overset{*}{\mathbf{e}}_3 &= - \sin\delta\,\bar{\mathbf{e}}_2 + \cos\delta\,\bar{\mathbf{e}}_3
\end{aligned}$$

$$\longrightarrow \quad \overset{*}{\mathbf{e}}_i = \tilde{\mathbf{R}}_3\,\underbrace{(\bar{\mathbf{R}}_1\,\bar{\mathbf{e}}_i)}_{\tilde{\mathbf{e}}_i} =: \bar{\mathbf{R}}\,\bar{\mathbf{e}}_i \qquad \text{with} \qquad \bar{\mathbf{R}} = \tilde{\mathbf{R}}_3\,\bar{\mathbf{R}}_1$$

(b) Inserting $\bar{\mathbf{e}}_i = \mathbf{R}_3\,\mathbf{e}_i$

$$\begin{aligned}
\overset{*}{\mathbf{e}}_1 &= \cos\psi\,(\cos\varphi\,\mathbf{e}_1 + \sin\varphi\,\mathbf{e}_2) + \sin\psi\,\cos\delta\,(-\sin\varphi\,\mathbf{e}_1 + \cos\varphi\,\mathbf{e}_2) + \sin\psi\,\sin\delta\,\mathbf{e}_3 \\
\overset{*}{\mathbf{e}}_2 &= -\sin\psi\,(\cos\varphi\,\mathbf{e}_1 + \sin\varphi\,\mathbf{e}_2) + \cos\psi\,\cos\delta\,(-\sin\varphi\,\mathbf{e}_1 + \cos\varphi\,\mathbf{e}_2) + \cos\psi\,\sin\delta\,\mathbf{e}_3 \\
\overset{*}{\mathbf{e}}_3 &= -\sin\delta\,(-\sin\varphi\,\mathbf{e}_1 + \cos\varphi\,\mathbf{e}_2) + \cos\delta\,\mathbf{e}_3
\end{aligned}$$

Result:

$$\begin{aligned}
\overset{*}{\mathbf{e}}_1 &= (\cos\psi\,\cos\varphi - \sin\psi\,\cos\delta\,\sin\varphi)\,\mathbf{e}_1 + \\
&\quad +(\cos\psi\,\sin\varphi + \sin\psi\,\cos\delta\,\cos\varphi)\,\mathbf{e}_2 + \sin\psi\,\sin\delta\,\mathbf{e}_3 \\
\overset{*}{\mathbf{e}}_2 &= (-\sin\psi\,\cos\varphi - \cos\psi\,\cos\delta\,\sin\varphi)\,\mathbf{e}_1 + \\
&\quad +(-\sin\psi\,\sin\varphi + \cos\psi\,\cos\delta\,\cos\varphi)\,\mathbf{e}_2 + \cos\psi\,\sin\delta\,\mathbf{e}_3 \\
\overset{*}{\mathbf{e}}_3 &= \sin\delta\,\sin\varphi\,\mathbf{e}_1 - \sin\delta\,\cos\varphi\,\mathbf{e}_2 + \cos\delta\,\mathbf{e}_3
\end{aligned}$$

$$\longrightarrow \quad \overset{*}{\mathbf{e}}_i = \bar{\mathbf{R}}\,\underbrace{(\mathbf{R}_3\,\mathbf{e}_i)}_{\bar{\mathbf{e}}_i} =: \mathbf{R}\,\mathbf{e}_i \qquad \text{with} \qquad \mathbf{R} = \bar{\mathbf{R}}\,\mathbf{R}_3 = \tilde{\mathbf{R}}_3\,\bar{\mathbf{R}}_1\,\mathbf{R}_3$$

Rotation tensors $\mathbf{R}$ and $\overset{*}{\mathbf{R}}$:

For the total rotation the following relation holds:

$$\begin{aligned}
\overset{*}{\mathbf{e}}_i &= (\tilde{\mathbf{R}}_3\,\bar{\mathbf{R}}_1\,\mathbf{R}_3)\,\mathbf{e}_i =: \mathbf{R}\,\mathbf{e}_i \\
&= (\tilde{\mathbf{R}}_3\,\bar{\mathbf{R}}_1)\,\underbrace{(\mathbf{R}_3\,\mathbf{e}_i)}_{\bar{\mathbf{e}}_i} = \tilde{\mathbf{R}}_3\,\underbrace{(\bar{\mathbf{R}}_1\,\bar{\mathbf{e}}_i)}_{\tilde{\mathbf{e}}_i} = \underbrace{\tilde{\mathbf{R}}_3\,\tilde{\mathbf{e}}_i}_{\overset{*}{\mathbf{e}}_i}
\end{aligned}$$

Furthermore,

$$\overset{*}{\mathbf{e}}_i = \mathbf{R}\,\mathbf{e}_i \quad \longrightarrow \quad \mathbf{e}_i = \mathbf{R}^T\,\overset{*}{\mathbf{e}}_i =: \overset{*}{\mathbf{R}}\,\overset{*}{\mathbf{e}}_i \quad \longrightarrow \quad \boxed{\overset{*}{\mathbf{R}} = \mathbf{R}^T}$$

Analogously to the previous considerations $\longrightarrow$ $\boxed{\overset{*}{R}_{ik} = (R_{ik})^T = R_{ki}}$

Description:

$$\mathbf{R} = \begin{bmatrix} \cos\psi\,\cos\varphi - \sin\psi\,\cos\delta\,\sin\varphi & -\sin\psi\,\cos\varphi - \cos\psi\,\cos\delta\,\sin\varphi & \sin\delta\,\sin\varphi \\ \cos\psi\,\sin\varphi + \sin\psi\,\cos\delta\,\cos\varphi & -\sin\psi\,\sin\varphi + \cos\psi\,\cos\delta\,\cos\varphi & -\sin\delta\,\cos\varphi \\ \sin\psi\,\sin\delta & \cos\psi\,\sin\delta & \cos\delta \end{bmatrix} \mathbf{e}_i \otimes \mathbf{e}_k$$

Combining rotation tensors with different basis systems:

Example: $\bar{\mathbf{R}} := \tilde{\mathbf{R}}_3\,\bar{\mathbf{R}}_1$

$$\overset{*}{\mathbf{e}}_i = \tilde{\mathbf{R}}_3\,\tilde{\mathbf{e}}_i = (\tilde{\mathbf{R}}_3\,\bar{\mathbf{R}}_1)\,\bar{\mathbf{e}}_i$$

$$\begin{aligned} \longrightarrow \quad \bar{\mathbf{R}} &= \tilde{R}_{3ik}\,(\tilde{\mathbf{e}}_i \otimes \tilde{\mathbf{e}}_k)\,\bar{R}_{1no}\,(\bar{\mathbf{e}}_n \otimes \bar{\mathbf{e}}_o) \\ &= \tilde{R}_{3ik}\,(\underbrace{\bar{\mathbf{R}}_1\,\bar{\mathbf{e}}_i \otimes \bar{\mathbf{R}}_1\,\bar{\mathbf{e}}_k}_{\bar{R}_{1si}\,\bar{\mathbf{e}}_s \otimes \bar{R}_{1tk}\,\bar{\mathbf{e}}_t})\,\bar{R}_{1no}\,(\bar{\mathbf{e}}_n \otimes \bar{\mathbf{e}}_o) \end{aligned}$$

$$\begin{aligned} \longrightarrow \quad \bar{\mathbf{R}} &= \bar{R}_{1si}\,\tilde{R}_{3ik}\,\bar{R}_{1tk}\,(\bar{\mathbf{e}}_s \otimes \bar{\mathbf{e}}_t)\,\bar{R}_{1no}\,(\bar{\mathbf{e}}_n \otimes \bar{\mathbf{e}}_o) \\ &= \bar{R}_{1si}\,\tilde{R}_{3ik}\,\bar{R}_{1tk}\,\bar{R}_{1no}\,\delta_{tn}\,(\bar{\mathbf{e}}_s \otimes \bar{\mathbf{e}}_o) \\ &= \underbrace{\bar{R}_{1si}\,\tilde{R}_{3ik}\,\bar{R}_{1tk}\,\bar{R}_{1to}}_{\bar{R}_{so}}\,(\bar{\mathbf{e}}_s \otimes \bar{\mathbf{e}}_o) \end{aligned}$$

Thus, the rotation tensor $\bar{\mathbf{R}}$ is given by

$$\bar{\mathbf{R}} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi\,\cos\delta & \cos\psi\,\cos\delta & -\sin\delta \\ \sin\psi\,\sin\delta & \cos\psi\,\sin\delta & \cos\delta \end{bmatrix} \bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_k$$

Remark: Concerning CARDANO angles, all partial rotations (e. g. $\mathbf{R} = \mathbf{R}_3\,\mathbf{R}_2\,\mathbf{R}_1$ with $\overset{*}{\mathbf{e}}_i = \mathbf{R}\,\mathbf{e}_i$) are carried out with respect to the same basis $\mathbf{e}_i$, i. e. the combination of the partial rotations is much easier.

Rotation around a fixed axis:

Remark: A rotation around 3 independent axes can also be described by a rotation around the resulting axis of rotation:

$\longrightarrow$ EULER-RODRIGUES representation of the spatial rotation

The EULER-RODRIGUES representation of the rotation is discussed later (see section 2.7).

Benjamin Olinde RODRIGUES (1795-1851) was a French mathematician, banker and social reformer.

2.5 Higher order tensors

Definition: An arbitrary nth order tensor is given by

$$\overset{n}{\mathbf{A}} \in \mathcal{V}^3 \otimes \mathcal{V}^3 \otimes \cdots \otimes \mathcal{V}^3 \quad (n \text{ times})$$
$$\text{with } \mathcal{V}^3 \otimes \mathcal{V}^3 \otimes \cdots \otimes \mathcal{V}^3 \quad : n\text{th order dyadic product space}$$

Remark: Usually, $n \geq 2$. However, there exist special cases for $n = 1$ (vector) and $n = 0$ (scalar).

General description of the linear mapping

Definition: A linear mapping is a "contracting product" (contraction) given by

$$\overset{n}{\mathbf{A}}\,\overset{s}{\mathbf{B}} = \overset{n-s}{\mathbf{C}} \quad \text{with} \quad n \geq s$$

Descriptive example on simple tensors:

$$\underbrace{(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} \otimes \mathbf{d})}_{\overset{4}{\mathbf{A}}}\,\underbrace{(\mathbf{e} \otimes \mathbf{f})}_{\mathbf{B}} = \underbrace{(\mathbf{c} \cdot \mathbf{e})\,(\mathbf{d} \cdot \mathbf{f})\,\mathbf{a} \otimes \mathbf{b}}_{\mathbf{C}}$$

Note: In the sense of the above definition of the linear mapping, the special case $n - s = 0$ yields a scalar and applies thus to the scalar or dot product.

Fundamental 4th order tensors

Remark: 4th order fundamental tensors are built by a dyadic product of 2nd order identity tensors and the corresponding independent transpositions.

One introduces:

$$\begin{aligned} \mathbf{I} \otimes \mathbf{I} \quad &= \quad (\mathbf{e}_i \otimes \mathbf{e}_i) \otimes (\mathbf{e}_j \otimes \mathbf{e}_j) \\ (\mathbf{I} \otimes \mathbf{I})^{\overset{23}{T}} \quad &= \quad \mathbf{e}_i \otimes \mathbf{e}_j \;\otimes\; \mathbf{e}_i \otimes \mathbf{e}_j \\ (\mathbf{I} \otimes \mathbf{I})^{\overset{24}{T}} \quad &= \quad \mathbf{e}_i \otimes \mathbf{e}_j \;\otimes\; \mathbf{e}_j \otimes \mathbf{e}_i \end{aligned}$$

with $(\,\cdot\,)^{\overset{ik}{T}}$: transposition, defined by the exchange of the ith and the kth basis system

Remark: Further transpositions of $\mathbf{I} \otimes \mathbf{I}$ do not lead to further independent tensors. The fundamental tensors from above exhibit the property

$$\overset{4}{\mathbf{A}} = \overset{4}{\mathbf{A}}{}^{T} \quad \text{with} \quad \overset{4}{\mathbf{A}}{}^{T} = (\overset{4}{\mathbf{A}}{}^{\overset{13}{T}})^{\overset{24}{T}}$$

Consequence: The 4th order fundamental tensors are symmetric (concerning an exchange of the first two and the second two basis systems).

Properties of 4th order fundamental tensors

(a) identical map

$$\begin{aligned}(\mathbf{I}\otimes\mathbf{I})^{\overset{23}{T}}\mathbf{A} &= (\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_i\otimes\mathbf{e}_j)\,a_{st}(\mathbf{e}_s\otimes\mathbf{e}_t)\\ &= a_{st}\,\delta_{is}\,\delta_{jt}\,(\mathbf{e}_i\otimes\mathbf{e}_j) = a_{ij}\,(\mathbf{e}_i\otimes\mathbf{e}_j) = \mathbf{A}\end{aligned}$$

$$\longrightarrow\quad \overset{4}{\mathbf{I}} := (\mathbf{I}\otimes\mathbf{I})^{\overset{23}{T}}\quad \text{is 4th order identity tensor}$$

(b) "transposing" map

$$\begin{aligned}(\mathbf{I}\otimes\mathbf{I})^{\overset{24}{T}}\mathbf{A} &= (\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_j\otimes\mathbf{e}_i)\,a_{st}\,(\mathbf{e}_s\otimes\mathbf{e}_t)\\ &= a_{st}\,\delta_{js}\,\delta_{it}\,(\mathbf{e}_i\otimes\mathbf{e}_j) = a_{ji}\,(\mathbf{e}_i\otimes\mathbf{e}_j) = \mathbf{A}^T\end{aligned}$$

(c) "tracing" map

$$\begin{aligned}(\mathbf{I}\otimes\mathbf{I})\,\mathbf{A} &= (\mathbf{e}_i\otimes\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_j)\,a_{st}\,(\mathbf{e}_s\otimes\mathbf{e}_t)\\ &= a_{st}\,\delta_{js}\,\delta_{jt}\,(\mathbf{e}_i\otimes\mathbf{e}_i) = a_{jj}\,(\mathbf{e}_i\otimes\mathbf{e}_i)\\ &= (\mathbf{A}\cdot\mathbf{I})\,\mathbf{I} = (\operatorname{tr}\mathbf{A})\,\mathbf{I}\end{aligned}$$

$$\text{with}\quad \mathbf{A}\cdot\mathbf{I} = a_{st}\,(\mathbf{e}_s\otimes\mathbf{e}_t)\cdot(\mathbf{e}_j\otimes\mathbf{e}_j) = a_{st}\,\delta_{sj}\,\delta_{tj} = a_{jj}$$

Specific 4th order tensors

Let $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$ be arbitrary 2nd order tensors. Then, a 4th order tensor $\overset{4}{\mathbf{A}}$ can be defined exhibiting the following properties:

$$\begin{aligned}\overset{4}{\mathbf{A}} &= (\mathbf{A}\otimes\mathbf{B})^{\overset{23}{T}} \qquad (*)\\ \overset{4}{\mathbf{A}}{}^{T} &= [(\mathbf{A}\otimes\mathbf{B})^{\overset{23}{T}}]^T = (\mathbf{A}^T\otimes\mathbf{B}^T)^{\overset{23}{T}}\\ \overset{4}{\mathbf{A}}{}^{-1} &= [(\mathbf{A}\otimes\mathbf{B})^{\overset{23}{T}}]^{-1} = (\mathbf{A}^{-1}\otimes\mathbf{B}^{-1})^{\overset{23}{T}}\end{aligned}$$

Furthermore, following relation holds:

$$\overset{4}{(\,\cdot\,)}{}^{T} = [\overset{4}{(\,\cdot\,)}{}^{\overset{13}{T}}]^{\overset{24}{T}}$$

From $(*)$, the following relations can be derived:

$$\begin{aligned}(\mathbf{A}\otimes\mathbf{B})^{\overset{23}{T}}(\mathbf{C}\otimes\mathbf{D})^{\overset{23}{T}} &= (\mathbf{AC}\otimes\mathbf{BD})^{\overset{23}{T}}\\ (\mathbf{A}\otimes\mathbf{B})^{\overset{23}{T}}(\mathbf{C}\otimes\mathbf{D}) &= (\mathbf{ACB}^T\otimes\mathbf{D})\\ (\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D})^{\overset{23}{T}} &= (\mathbf{A}\otimes\mathbf{C}^T\mathbf{BD})\end{aligned}$$

and

$$\begin{aligned}(\mathbf{A}\otimes\mathbf{B})^{\overset{23}{T}}\mathbf{C} &= \mathbf{ACB}^T\\ (\mathbf{A}\otimes\mathbf{B})^{\overset{23}{T}}\mathbf{v} &= [\mathbf{A}\otimes(\mathbf{B}\,\mathbf{v})]^{\overset{23}{T}}\end{aligned}$$

Defining a 4th order tensor $\overset{4}{\mathbf{B}}$ with the properties

$$\begin{aligned}
\overset{4}{\mathbf{B}} &= (\mathbf{A} \otimes \mathbf{B})^{\overset{24}{T}} = [(\mathbf{A} \otimes \mathbf{B})^{\overset{13}{T}}]^T \\
\overset{4}{\mathbf{B}}{}^T &= [(\mathbf{A} \otimes \mathbf{B})^{\overset{24}{T}}]^T = (\mathbf{B} \otimes \mathbf{A})^{\overset{24}{T}} \\
\overset{4}{\mathbf{B}}{}^{-1} &= [(\mathbf{A} \otimes \mathbf{B})^{\overset{24}{T}}]^{-1} = (\mathbf{B}^{T-1} \otimes \mathbf{A}^{T-1})^{\overset{24}{T}}
\end{aligned}$$

it can be shown that

$$\begin{aligned}
(\mathbf{A} \otimes \mathbf{B})^{\overset{24}{T}}(\mathbf{C} \otimes \mathbf{D})^{\overset{24}{T}} &= (\mathbf{A}\mathbf{D}^T \otimes \mathbf{B}^T\mathbf{C})^{\overset{23}{T}} \\
(\mathbf{A} \otimes \mathbf{B})^{\overset{23}{T}}(\mathbf{C} \otimes \mathbf{D})^{\overset{24}{T}} &= (\mathbf{A}\mathbf{C} \otimes \mathbf{D}\mathbf{B}^T)^{\overset{24}{T}} \\
(\mathbf{A} \otimes \mathbf{B})^{\overset{24}{T}}(\mathbf{C} \otimes \mathbf{D})^{\overset{23}{T}} &= (\mathbf{A}\mathbf{D} \otimes \mathbf{C}^T\mathbf{B})^{\overset{24}{T}} \\
(\mathbf{A} \otimes \mathbf{B})^{\overset{24}{T}}(\mathbf{C} \otimes \mathbf{D}) &= (\mathbf{A}\mathbf{C}^T\mathbf{B} \otimes \mathbf{D}) \\
(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D})^{\overset{24}{T}} &= (\mathbf{A} \otimes \mathbf{D}\mathbf{B}^T\mathbf{C})
\end{aligned}$$

and

$$(\mathbf{A} \otimes \mathbf{B})^{\overset{24}{T}}\mathbf{C} = \mathbf{A}\mathbf{C}^T\mathbf{B}$$

Furthermore, the following relation holds:

$$(\overset{4}{\mathbf{C}}\overset{4}{\mathbf{D}})^T = \overset{4}{\mathbf{D}}{}^T\overset{4}{\mathbf{C}}{}^T$$

where $\overset{4}{\mathbf{C}}$ and $\overset{4}{\mathbf{D}}$ are arbitrary 4th order tensors.

Higher order tensors and incomplete mappings

If higher order tensors are applied to other tensors in the sense of incomplete mappings, one has to know how many of the basis vectors have to be linked by scalar products. Therefore, an underlined superscript $(\cdot)^{\underline{i}}$ indicates the order of the desired result after the tensor operation has been carried out.

Examples in basis notation:

$$\begin{aligned}
(\overset{4}{\mathbf{A}}\,\overset{3}{\mathbf{B}})^{\underline{3}} &= [a_{ijkl}\,(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l)\,b_{mno}\,(\mathbf{e}_m \otimes \mathbf{e}_n \otimes \mathbf{e}_o)]^{\underline{3}} \\
&= a_{ijkl}\,b_{mno}\,\delta_{km}\,\delta_{ln}\,(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_o) \\
(\mathbf{A}\,\overset{3}{\mathbf{B}})^{\underline{1}} &= [a_{ij}\,(\mathbf{e}_i \otimes \mathbf{e}_j)\,b_{mno}\,(\mathbf{e}_m \otimes \mathbf{e}_n \otimes \mathbf{e}_o)]^{\underline{1}} \\
&= a_{ij}\,b_{mno}\,\delta_{im}\,\delta_{jn}\,\mathbf{e}_o
\end{aligned}$$

Note: Note in passing that the incomplete mapping (contraction) is governed by scalar products of a sufficient number of inner basis systems.
Furthermore, the tensor product of 2nd order tensors can also be understood as an incomplete mapping by $\mathbf{AB} = (\mathbf{AB})^{\underline{2}}$.

2.6 Fundamental tensor of 3rd order (Ricci permutation tensor)

Remark: The fundamental tensor of 3rd order is introduced in the context of the "outer product" (e. g. vector product between vectors).

Definition: The fundamental tensor $\overset{3}{\mathbf{E}}$ satisfies the rule

$$\mathbf{u} \times \mathbf{v} = \overset{3}{\mathbf{E}} \; (\mathbf{u} \otimes \mathbf{v})$$

Introduction of $\overset{3}{\mathbf{E}}$ in basis notation:

There is

$$\overset{3}{\mathbf{E}} = \mathrm{e}_{ijk} \, (\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k)$$

with the "permutation symbol" e_{ijk}

$$e_{ijk} = \begin{cases} 1 & : \text{even permutation} \\ -1 & : \text{odd permutation} \\ 0 & : \text{double indexing} \end{cases} \longrightarrow \begin{cases} \mathrm{e}_{123} = \mathrm{e}_{231} = \mathrm{e}_{312} = 1 \\ \mathrm{e}_{321} = \mathrm{e}_{213} = \mathrm{e}_{132} = -1 \\ \text{all remaining } \mathrm{e}_{ijk} \text{ vanish} \end{cases}$$

Application of $\overset{3}{\mathbf{E}}$ to the vector product of vectors:

From the above definition,

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \overset{3}{\mathbf{E}} \; (\mathbf{u} \otimes \mathbf{v}) \\ &= \mathrm{e}_{ijk} \, (\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k) \, (u_s \, \mathbf{e}_s \otimes v_t \, \mathbf{e}_t) \\ &= \mathrm{e}_{ijk} \, u_s \, v_t \, \delta_{js} \, \delta_{kt} \, \mathbf{e}_i = \mathrm{e}_{ijk} \, u_j \, v_k \, \mathbf{e}_i \\ &= (u_2 \, v_3 - u_3 \, v_2) \, \mathbf{e}_1 + (u_3 \, v_1 - u_1 \, v_3) \, \mathbf{e}_2 + (u_1 \, v_2 - u_2 \, v_1) \, \mathbf{e}_3 \end{aligned}$$

Comparison with the computation by use of the matrix notation, cf. page 5

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \cdots \quad \text{q. e. d.}$$

Identities for $\overset{3}{\mathbf{E}}$:

Scalar product and incomplete mapping of two Ricci tensors yield a scalar and 2nd or 4th order objects

$$\overset{3}{\mathbf{E}} \cdot \overset{3}{\mathbf{E}} = 6 \, , \quad (\overset{3}{\mathbf{E}} \, \overset{3}{\mathbf{E}})^{\underline{2}} = 2 \, \mathbf{I} \, , \quad (\overset{3}{\mathbf{E}} \, \overset{3}{\mathbf{E}})^{\underline{4}} = (\mathbf{I} \otimes \mathbf{I})^{\overset{23}{T}} - (\mathbf{I} \otimes \mathbf{I})^{\overset{24}{T}}$$

2.7 The axial vector

Remark: The axial vector (pseudo vector) can be used for the description of rotations (rotation vector).

Definition: The axial vector $\overset{A}{\mathbf{t}}$ is associated with the skew-symmetric part $\operatorname{skw}\mathbf{T}$ of an arbitrary tensor $\mathbf{T} \in \mathcal{V}^3 \otimes \mathcal{V}^3$ via

$$\overset{A}{\mathbf{t}} := \tfrac{1}{2} \overset{3}{\mathbf{E}} \ \mathbf{T}^T$$

One calculates,

$$\begin{aligned} \overset{A}{\mathbf{t}} &= \tfrac{1}{2}\, \mathrm{e}_{ijk} \,(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k)\, t_{st}\,(\mathbf{e}_t \otimes \mathbf{e}_s) \\ &= \tfrac{1}{2}\, \mathrm{e}_{ijk}\, t_{st}\, \delta_{jt}\, \delta_{ks}\, \mathbf{e}_i = \tfrac{1}{2}\, \mathrm{e}_{ijk}\, t_{kj}\, \mathbf{e}_i \\ &= \tfrac{1}{2}\,[(t_{32} - t_{23})\,\mathbf{e}_1 + (t_{13} - t_{31})\,\mathbf{e}_2 + (t_{21} - t_{12})\,\mathbf{e}_3] \end{aligned}$$

It follows from 2.3 (b)

$$\mathbf{T} = \operatorname{sym}\mathbf{T} + \operatorname{skw}\mathbf{T}$$

Thus, the axial vector of $\mathbf{T}$ is given by

$$\begin{aligned} \overset{A}{\mathbf{t}} &= \tfrac{1}{2} \overset{3}{\mathbf{E}} \,(\operatorname{sym}\mathbf{T} + \operatorname{skw}\mathbf{T})^T \\ &= \tfrac{1}{2} \overset{3}{\mathbf{E}} \,(\operatorname{skw}\mathbf{T}^T) = -\tfrac{1}{2} \overset{3}{\mathbf{E}} \,(\operatorname{skw}\mathbf{T}) \end{aligned}$$

Remark: A symmetric tensor has no axial vector.

Axial vector and linear mapping:

The following relation holds:

$$(\operatorname{skw}\mathbf{T})\,\mathbf{v} = \overset{A}{\mathbf{t}} \times \mathbf{v} \quad \forall\, \mathbf{v} \in \mathcal{V}^3$$

Axial vector and the vector product of tensors:

Definition: The vector product of 2 tensors $\{\mathbf{T}, \mathbf{S}\} \in \mathcal{V}^3 \otimes \mathcal{V}^3$ satisfies

$$\mathbf{S} \times \mathbf{T} = \overset{3}{\mathbf{E}} \ (\mathbf{S}\,\mathbf{T}^T)$$

Remark: The vector or cross product of two tensors yields a vector.

In comparison with the definition of the axial vector follows

$$\mathbf{I} \times \mathbf{T} = \overset{3}{\mathbf{E}} \ \mathbf{T}^T = 2 \ \overset{A}{\mathbf{t}}$$

Furthermore, the vector product of two tensors yields

$$\mathbf{S} \times \mathbf{T} = -\mathbf{T} \times \mathbf{S}$$

Axial vector and outer tensor product of vector and tensor:

Definition: The outer tensor product of a vector $\mathbf{u} \in \mathcal{V}^3$ and a tensor $\mathbf{T} \in \mathcal{V}^3 \otimes \mathcal{V}^3$ satisfies

$$(\mathbf{u} \times \mathbf{T})\,\mathbf{v} = \mathbf{u} \times (\mathbf{T}\,\mathbf{v})\ ; \quad \mathbf{v} \in \mathcal{V}^3$$

Remark: The outer tensor product of vector and tensor yields a tensor.

The following relations hold:

$$\mathbf{u} \times \mathbf{T} = -(\mathbf{u} \times \mathbf{T})^T = -\mathbf{T} \times \mathbf{u}$$

$\longrightarrow$ i. e. $\mathbf{u} \times \mathbf{T}$ is skew-symmetric

$$\mathbf{u} \times \mathbf{T} = [\overset{3}{\mathbf{E}}\ (\mathbf{u} \otimes \mathbf{T})]^{\underline{2}}$$

with $(\,\cdot\,)^{\underline{2}}$: "incomplete" linear mapping (contraction) resulting in a 2nd order tensor.

Evaluation in basis notation leads to

$$\begin{aligned} \mathbf{u} \times \mathbf{T} &= [(\mathrm{e}_{ijk}\,\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k)\,(u_r\,\mathbf{e}_r \otimes t_{st}\,\mathbf{e}_s \otimes \mathbf{e}_t)]^{\underline{2}} \\ &= \mathrm{e}_{ijk}\,u_r\,t_{st}\,\delta_{jr}\,\delta_{ks}\,(\mathbf{e}_i \otimes \mathbf{e}_t) \\ &= \mathrm{e}_{ijk}\,u_j\,t_{kt}\,(\mathbf{e}_i \otimes \mathbf{e}_t) \end{aligned}$$

In particular, if $\mathbf{T} \equiv \mathbf{I}$, the following relation holds:

$$\mathbf{u} \times \mathbf{I} = [\overset{3}{\mathbf{E}}\ (\mathbf{u} \otimes \mathbf{I})]^{\underline{2}} = \mathrm{e}_{ijk}\,u_j\,\delta_{kt}\,(\mathbf{e}_i \otimes \mathbf{e}_t) = \mathrm{e}_{ijt}\,u_j\,(\mathbf{e}_i \otimes \mathbf{e}_t)$$

Furthermore, for the special tensor $\mathbf{u} \times \mathbf{I}$ follows

$$\overset{3}{\mathbf{E}}\ (\mathbf{u} \times \mathbf{I}) = -2\,\mathbf{u}$$

$$\longrightarrow \quad \mathbf{u} = -\tfrac{1}{2}\,\overset{3}{\mathbf{E}}\ (\mathbf{u} \times \mathbf{I}) = \tfrac{1}{2}\,\overset{3}{\mathbf{E}}\ (\mathbf{u} \times \mathbf{I})^T$$

Consequence: In the tensor $\mathbf{u} \times \mathbf{I}$, the vector $\mathbf{u}$ is already the corresponding axial vector.

Finally, the following relation holds:

$$\mathbf{u} \times \mathbf{I} = -\,\overset{3}{\mathbf{E}}\ \mathbf{u}$$

$$\longrightarrow \quad \overset{3}{\mathbf{E}}\ (\mathbf{u} \times \mathbf{I}) = -\,\overset{3}{\mathbf{E}}\ (\overset{3}{\mathbf{E}}\ \mathbf{u}) = -(\overset{3}{\mathbf{E}}\,\overset{3}{\mathbf{E}})^{\underline{2}}\,\mathbf{u} \overset{!}{=} -2\,\mathbf{u}$$

$$\text{with} \quad (\overset{3}{\mathbf{E}}\,\overset{3}{\mathbf{E}})^{\underline{2}} = 2\,\mathbf{I}$$

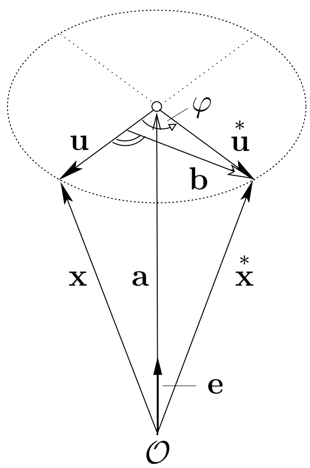
Some additional rules:

$$(\mathbf{a} \times \mathbf{b}) \otimes \mathbf{c} = \mathbf{a} \times (\mathbf{b} \otimes \mathbf{c})$$

$$(\mathbf{I} \times \mathbf{T}) \cdot \mathbf{w} = \mathbf{T} \cdot \mathbf{\Omega} \quad \text{with} \quad \mathbf{\Omega} = \mathbf{w} \times \mathbf{I}$$

Application to the tensor product of vector and tensor

Rotation around a fixed spatial axis



Rotation of $\mathbf{x}$ around axis $\mathbf{e}$

$$\overset{*}{\mathbf{x}} = \mathbf{a} + \overset{*}{\mathbf{u}} = \mathbf{a} + C_1\,\mathbf{u} + \mathbf{b}$$

$$\text{with} \quad \begin{cases} \mathbf{a} = (\mathbf{x} \cdot \mathbf{e})\,\mathbf{e} \\ \mathbf{u} = \mathbf{x} - \mathbf{a} \\ \mathbf{b} = C_2\,(\mathbf{e} \times \mathbf{x}) \end{cases}$$

$$\text{and} \quad \boldsymbol{\varphi} = \varphi\,\mathbf{e}\,; \quad |\mathbf{e}| = 1$$

Determination of the constants C_1 and C_2:

(a) For the angle between $\mathbf{u}$ and $\overset{*}{\mathbf{u}}$, the following relation holds

$$\cos\varphi = \frac{\mathbf{u} \cdot \overset{*}{\mathbf{u}}}{|\mathbf{u}|\,|\overset{*}{\mathbf{u}}|} \qquad \text{with} \quad |\mathbf{u}| = |\overset{*}{\mathbf{u}}|$$

Furthermore, one calculates

$$\mathbf{u} \cdot \overset{*}{\mathbf{u}} = \mathbf{u} \cdot (C_1\,\mathbf{u} + \mathbf{b}) = C_1\,\mathbf{u} \cdot \mathbf{u} + \underbrace{\mathbf{u} \cdot \mathbf{b}}_{=0,\ \text{da}\ \mathbf{u} \perp \mathbf{b}} = C_1\,|\mathbf{u}|^2$$

Thus,

$$\cos\varphi = \frac{C_1\,|\mathbf{u}|^2}{|\mathbf{u}|^2} = C_1 \quad \longrightarrow \quad C_1 = \cos\varphi$$

(b) For the angle between $\mathbf{b}$ and $\overset{*}{\mathbf{u}}$, the following relation holds

$$\cos(90^\circ - \varphi) = \sin\varphi = \frac{\mathbf{b} \cdot \overset{*}{\mathbf{u}}}{|\mathbf{b}|\,|\overset{*}{\mathbf{u}}|}$$

Furthermore, one calculates

$$\mathbf{b} \cdot \overset{*}{\mathbf{u}} = \mathbf{b} \cdot (C_1\,\mathbf{u} + \mathbf{b}) = C_1\,\underbrace{\mathbf{b} \cdot \mathbf{u}}_{=0,\ \text{da}\ \mathbf{u} \perp \mathbf{b}} + \mathbf{b} \cdot \mathbf{b} = |\mathbf{b}|^2$$

and

$$|\mathbf{b}| = C_2\,|\mathbf{e} \times \mathbf{x}| = C_2\,\underbrace{|\mathbf{e}|}_{1}\,\underbrace{|\mathbf{x}|\sin\sphericalangle(\mathbf{e}\,;\,\mathbf{x})}_{|\mathbf{u}|} = C_2\,|\mathbf{u}|$$

leading to

$$\sin\varphi = \frac{|\mathbf{b}|^2}{|\mathbf{b}|\,|\mathbf{u}|} = \frac{|\mathbf{b}|}{|\mathbf{u}|} = \frac{C_2\,|\mathbf{u}|}{|\mathbf{u}|} = C_2 \quad \longrightarrow \quad C_2 = \sin\varphi$$

Thus, $\overset{*}{\mathbf{x}}$ is given by

$$\overset{*}{\mathbf{x}} = (\mathbf{x}\cdot\mathbf{e})\,\mathbf{e} + \cos\varphi\,[\mathbf{x} - (\mathbf{x}\cdot\mathbf{e})\,\mathbf{e}] + \sin\varphi\,(\mathbf{e}\times\mathbf{x})$$

Determination of the rotation tensor R:

For the tensor product of vector and tensor, the following relation holds:

$$(\mathbf{e}\times\mathbf{I})\,\mathbf{x} = \mathbf{e}\times(\mathbf{I}\,\mathbf{x}) = \mathbf{e}\times\mathbf{x}$$

Thus,

$$\overset{*}{\mathbf{x}} = (\mathbf{e}\otimes\mathbf{e})\,\mathbf{x} + \cos\varphi\,(\mathbf{I} - \mathbf{e}\otimes\mathbf{e})\,\mathbf{x} + \sin\varphi\,(\mathbf{e}\times\mathbf{I})\,\mathbf{x} \overset{!}{=} \mathbf{R}\,\mathbf{x}$$

$$\longrightarrow \quad \boxed{\mathbf{R} = \mathbf{e}\otimes\mathbf{e} + \cos\varphi\,(\mathbf{I} - \mathbf{e}\otimes\mathbf{e}) + \sin\varphi\,(\mathbf{e}\times\mathbf{I})} \qquad (*)$$

Remark: $(*)$ is the EULER-RODRIGUES form of the spatial rotation.

Example: Rotation with φ_3 around the $\mathbf{e}_3$ axis

$$\mathbf{R} = \mathbf{R}_3 = \mathbf{e}_3\otimes\mathbf{e}_3 + \cos\,\varphi_3\,(\mathbf{I} - \mathbf{e}_3\otimes\mathbf{e}_3) + \sin\,\varphi_3\,(\mathbf{e}_3\times\mathbf{I})$$

The following relation holds:

$$\begin{aligned}
\mathbf{e}_3\times\mathbf{I} &= [\overset{3}{\mathbf{E}}\,(\mathbf{e}_3\otimes\mathbf{I})]^{\underline{2}} \\
&= [\mathrm{e}_{ijk}\,(\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k)\,(\mathbf{e}_3\otimes\mathbf{e}_l\otimes\mathbf{e}_l)]^{\underline{2}} \\
&= \mathrm{e}_{ijk}\,\delta_{j3}\,\delta_{kl}\,(\mathbf{e}_i\otimes\mathbf{e}_l) = \mathrm{e}_{i3l}\,(\mathbf{e}_i\otimes\mathbf{e}_l) \\
&= \mathbf{e}_2\otimes\mathbf{e}_1 - \mathbf{e}_1\otimes\mathbf{e}_2
\end{aligned}$$

Thus, one obtains

$$\begin{aligned}
\mathbf{R}_3 &= \mathbf{e}_3\otimes\mathbf{e}_3 + \cos\varphi_3\,(\mathbf{e}_1\otimes\mathbf{e}_1 + \mathbf{e}_2\otimes\mathbf{e}_2) + \sin\varphi_3\,(\mathbf{e}_2\otimes\mathbf{e}_1 - \mathbf{e}_1\otimes\mathbf{e}_2) \\
&= R_{3ij}\,(\mathbf{e}_i\otimes\mathbf{e}_j)
\end{aligned}$$

$$\text{with} \qquad R_{3ij} = \begin{bmatrix} \cos\varphi_3 & -\sin\varphi_3 & 0 \\ \sin\varphi_3 & \cos\varphi_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{q. e. d.}$$

2.8 The outer tensor product of tensors

Definition: The outer tensor product of tensors (double cross product) is defined via

$$(\mathbf{A} \divideontimes \mathbf{B})(\mathbf{u}_1 \times \mathbf{u}_2) := \mathbf{A}\mathbf{u}_1 \times \mathbf{B}\mathbf{u}_2 - \mathbf{A}\mathbf{u}_2 \times \mathbf{B}\mathbf{u}_1$$

As a direct consequence, one finds

$$\mathbf{A} \divideontimes \mathbf{B} = \mathbf{B} \divideontimes \mathbf{A}$$

Furthermore, the following relations hold:

$$\begin{aligned}
(\mathbf{A} \divideontimes \mathbf{B})^T &= \mathbf{A}^T \divideontimes \mathbf{B}^T \\
(\mathbf{A} \divideontimes \mathbf{B})\,(\mathbf{C} \divideontimes \mathbf{D}) &= (\mathbf{A}\,\mathbf{C} \divideontimes \mathbf{B}\,\mathbf{D}) + (\mathbf{A}\,\mathbf{D} \divideontimes \mathbf{B}\,\mathbf{C}) \\
(\mathbf{I} \divideontimes \mathbf{I}) &= 2\,\mathbf{I} \\
(\mathbf{a} \otimes \mathbf{b}) \divideontimes (\mathbf{c} \otimes \mathbf{d}) &= (\mathbf{a} \times \mathbf{c}) \otimes (\mathbf{b} \times \mathbf{d}) \\
(\mathbf{A} \divideontimes \mathbf{B}) \cdot \mathbf{C} &= (\mathbf{B} \divideontimes \mathbf{C}) \cdot \mathbf{A} = (\mathbf{C} \divideontimes \mathbf{A}) \cdot \mathbf{B}
\end{aligned}$$

From the above definition, it is easily proven that

$$[(\mathbf{A} \divideontimes \mathbf{B}) \cdot \mathbf{C}][(\mathbf{u}_1 \times \mathbf{u}_2) \cdot \mathbf{u}_3] = e_{ijk}\,(\mathbf{A}\mathbf{u}_i \times \mathbf{B}\mathbf{u}_j) \cdot \mathbf{C}\mathbf{u}_k$$

The outer tensor product in basis notation reads

$$\begin{aligned}
\mathbf{A} \divideontimes \mathbf{B} \quad &= a_{ik}\,(\mathbf{e}_i \otimes \mathbf{e}_k) \divideontimes b_{no}\,(\mathbf{e}_n \otimes \mathbf{e}_o) \\
&= a_{ik}\,b_{no}\,(\mathbf{e}_i \times \mathbf{e}_n) \otimes (\mathbf{e}_k \times \mathbf{e}_o) \\
\text{with} \quad & \begin{cases} \mathbf{e}_i \times \mathbf{e}_n = \overset{3}{\mathbf{E}}\,(\mathbf{e}_i \otimes \mathbf{e}_n) = e_{inj}\,\mathbf{e}_j \\ \mathbf{e}_k \times \mathbf{e}_o = \overset{3}{\mathbf{E}}\,(\mathbf{e}_k \otimes \mathbf{e}_o) = e_{kop}\,\mathbf{e}_p \end{cases} \\
\longrightarrow \quad & \mathbf{A} \divideontimes \mathbf{B} = a_{ik}\,b_{no}\,e_{inj}\,e_{kop}\,(\mathbf{e}_j \otimes \mathbf{e}_p)
\end{aligned}$$

Furthermore, it follows that

$$\begin{aligned}
\mathbf{A} \divideontimes \mathbf{I} &= (\mathbf{A} \cdot \mathbf{I})\,\mathbf{I} - \mathbf{A}^T \\
\mathbf{A} \divideontimes \mathbf{B} &= (\mathbf{A} \cdot \mathbf{I})\,(\mathbf{B} \cdot \mathbf{I})\,\mathbf{I} - (\mathbf{A}^T \cdot \mathbf{B})\,\mathbf{I} - (\mathbf{A} \cdot \mathbf{I})\,\mathbf{B}^T - \\
&\quad -(\mathbf{B} \cdot \mathbf{I})\,\mathbf{A}^T + \mathbf{A}^T\,\mathbf{B}^T + \mathbf{B}^T\,\mathbf{A}^T \\
(\mathbf{A} \divideontimes \mathbf{B}) \cdot \mathbf{C} &= (\mathbf{A} \cdot \mathbf{I})\,(\mathbf{B} \cdot \mathbf{I})\,(\mathbf{C} \cdot \mathbf{I}) - (\mathbf{A} \cdot \mathbf{I})\,(\mathbf{B}^T \cdot \mathbf{C}) - (\mathbf{B} \cdot \mathbf{I})\,(\mathbf{A}^T \cdot \mathbf{C}) - \\
&\quad -(\mathbf{C} \cdot \mathbf{I})\,(\mathbf{A}^T \cdot \mathbf{B}) + (\mathbf{A}^T\,\mathbf{B}^T) \cdot \mathbf{C} + (\mathbf{B}^T\,\mathbf{A}^T) \cdot \mathbf{C}
\end{aligned}$$

The cofactor, the adjoint tensor and the determinant:

The following relations hold:

$$\operatorname{cof}\mathbf{A} = \tfrac{1}{2}\,\mathbf{A} \mathbin{⨳} \mathbf{A} =: \overset{+}{\mathbf{A}}\,, \quad \operatorname{adj}\mathbf{A} = (\operatorname{cof}\mathbf{A})^T$$

$$\det\mathbf{A} = \tfrac{1}{6}\,(\mathbf{A} \mathbin{⨳} \mathbf{A}) \cdot \mathbf{A} = \det|a_{ik}| = \frac{(\mathbf{A}\mathbf{u}_1 \times \mathbf{A}\mathbf{u}_2) \cdot \mathbf{A}\mathbf{u}_3}{(\mathbf{u}_1 \times \mathbf{u}_2) \cdot \mathbf{u}_3}$$

$$= \tfrac{1}{6}\,(\mathbf{A} \cdot \mathbf{I})^3 - \tfrac{1}{2}(\mathbf{A} \cdot \mathbf{I})(\mathbf{A}\mathbf{A} \cdot \mathbf{I}) + \tfrac{1}{3}(\mathbf{A}\mathbf{A}\mathbf{A} \cdot \mathbf{I}) \quad (*)$$

In basis notation, the cofactor of $\mathbf{A}$ reads

$$\operatorname{cof}\mathbf{A} = \tfrac{1}{2}\,(a_{ik}\, a_{no}\, e_{inj}\, e_{kop})\,(\mathbf{e}_j \otimes \mathbf{e}_p) = \overset{+}{a}_{jp}\ (\mathbf{e}_j \otimes \mathbf{e}_p)$$

Remark: The coefficient matrix $\overset{+}{a}_{jp}$ of the cofactor $\operatorname{cof}\mathbf{A}$ contains at each position $(\,\cdot\,)_{jp}$ the corresponding subdeterminant of $\mathbf{A}$

$$\overset{+}{a}_{11} = a_{22}\, a_{33} - a_{23}\, a_{32} \quad \text{etc.}$$

The inverse tensor:

The following relation holds:

$$\mathbf{A}^{-1} = \frac{\operatorname{cof}\mathbf{A}^T}{\det\mathbf{A}}\,; \quad \mathbf{A}^{-1} \text{ exists if } \det\mathbf{A} \neq 0$$

Rules for the cofactor, the determinant and the inverse tensor:

$$\det(\mathbf{A}\,\mathbf{B}) = \det\mathbf{A}\det\mathbf{B}$$

$$\det(\alpha\,\mathbf{A}) = \alpha^3 \det\mathbf{A}$$

$$\det\mathbf{I} = 1$$

$$\det\mathbf{A}^T = \det\mathbf{A}$$

$$\det\overset{+}{\mathbf{A}} = (\det\mathbf{A})^2$$

$$\det\mathbf{A}^{-1} = (\det\mathbf{A})^{-1}$$

$$\det(\mathbf{A} + \mathbf{B}) = \det\mathbf{A} + \overset{+}{\mathbf{A}} \cdot \mathbf{B} + \mathbf{A} \cdot \overset{+}{\mathbf{B}} + \det\mathbf{B}$$

$$\overset{+}{(\mathbf{A}\,\mathbf{B})} = \overset{+}{\mathbf{A}}\,\overset{+}{\mathbf{B}}$$

$$(\overset{+}{\mathbf{A}})^T = \overset{+}{(\mathbf{A}^T)}$$

2.9 The eigenvalue problem and the invariants of tensors

Definition: The eigenvalue problem of an arbitrary 2nd order tensor $\mathbf{A}$ is given by

$$(\mathbf{A} - \gamma_{\mathbf{A}}\,\mathbf{I})\,\mathbf{a} = \mathbf{0}\,, \quad \text{where} \begin{cases} \gamma_{\mathbf{A}} & : \text{eigenvalue} \\ \mathbf{a} & : \text{eigenvector} \end{cases}$$

Formal solution for $\mathbf{a}$ yields

$$\mathbf{a} = (\mathbf{A} - \gamma_{\mathbf{A}}\,\mathbf{I})^{-1}\,\mathbf{0} = \frac{\operatorname{cof}(\mathbf{A} - \gamma_{\mathbf{A}}\,\mathbf{I})^T}{\det(\mathbf{A} - \gamma_{\mathbf{A}}\,\mathbf{I})}\,\mathbf{0}$$

Consequence: Non-trivial solution for $\mathbf{a}$ only if the characteristic equation is fulfilled, such that

$$\det(\mathbf{A} - \gamma_{\mathbf{A}}\,\mathbf{I}) = 0$$

With the determinant rule

$$\begin{aligned} \det(\mathbf{A} + \mathbf{B}) &= \tfrac{1}{6}\,[(\mathbf{A} + \mathbf{B}) \divideontimes (\mathbf{A} + \mathbf{B})] \cdot (\mathbf{A} + \mathbf{B}) \\ &= \tfrac{1}{6}\,(\mathbf{A} \divideontimes \mathbf{A}) \cdot \mathbf{A} + \tfrac{1}{6}\,(\mathbf{A} \divideontimes \mathbf{A}) \cdot \mathbf{B} + \tfrac{1}{3}\,(\mathbf{A} \divideontimes \mathbf{B}) \cdot \mathbf{A} + \\ &\quad + \tfrac{1}{3}\,(\mathbf{A} \divideontimes \mathbf{B}) \cdot \mathbf{B} + \tfrac{1}{6}\,(\mathbf{B} \divideontimes \mathbf{B}) \cdot \mathbf{A} + \tfrac{1}{6}\,(\mathbf{B} \divideontimes \mathbf{B}) \cdot \mathbf{B} \\ &= \det \mathbf{A} + \overset{+}{\mathbf{A}} \cdot \mathbf{B} + \mathbf{A} \cdot \overset{+}{\mathbf{B}} + \det \mathbf{B} \end{aligned}$$

follows

$$\begin{aligned} \det(\mathbf{A} - \gamma_{\mathbf{A}}\,\mathbf{I}) &= \det \mathbf{A} + \overset{+}{\mathbf{A}} \cdot (-\gamma_{\mathbf{A}}\,\mathbf{I}) + \mathbf{A} \cdot (-\gamma_{\mathbf{A}}\,\mathbf{I})^{+} + \det(-\gamma_{\mathbf{A}}\,\mathbf{I}) \\ &= \det \mathbf{A} - \gamma_{\mathbf{A}}\,\tfrac{1}{2}\,(\mathbf{A} \divideontimes \mathbf{A}) \cdot \mathbf{I} + \gamma_{\mathbf{A}}^2\,\tfrac{1}{2}\,\mathbf{A} \cdot (\mathbf{I} \divideontimes \mathbf{I}) - \gamma_{\mathbf{A}}^3 \det \mathbf{I} = 0 \end{aligned}$$

With the abreviations

$$\begin{aligned} I_{\mathbf{A}} &= \tfrac{1}{2}\,(\mathbf{A} \divideontimes \mathbf{I}) \cdot \mathbf{I} \\ II_{\mathbf{A}} &= \tfrac{1}{2}\,(\mathbf{A} \divideontimes \mathbf{A}) \cdot \mathbf{I} \\ III_{\mathbf{A}} &= \tfrac{1}{6}\,(\mathbf{A} \divideontimes \mathbf{A}) \cdot \mathbf{A} \end{aligned}$$

the characteristic equation can be simplified to

$$\det(\mathbf{A} - \gamma_{\mathbf{A}}\,\mathbf{I}) = III_{\mathbf{A}} - \gamma_{\mathbf{A}}\,II_{\mathbf{A}} + \gamma_{\mathbf{A}}^2\,I_{\mathbf{A}} - \gamma_{\mathbf{A}}^3 = 0$$

Remark: The abbreviations $I_{\mathbf{A}}$, $II_{\mathbf{A}}$ and $III_{\mathbf{A}}$ are the **three scalar principal invariants** of a tensor $\mathbf{A}$ which play an important role in the field of continuum mechanics.

Alternative representations of the principal invariants

Scalar-product representation:

$$\begin{aligned} I_{\mathbf{A}} &= \mathbf{A} \cdot \mathbf{I} = \operatorname{tr} \mathbf{A} \\ II_{\mathbf{A}} &= \tfrac{1}{2}\,(I_{\mathbf{A}}^2 - \mathbf{A}\,\mathbf{A} \cdot \mathbf{I}) = \tfrac{1}{2}\,[(\operatorname{tr} \mathbf{A})^2 - \operatorname{tr}(\mathbf{A}\,\mathbf{A})] \\ III_{\mathbf{A}} &= \tfrac{1}{6}\,I_{\mathbf{A}}^3 - \tfrac{1}{2}\,I_{\mathbf{A}}^2\,(\mathbf{A}\,\mathbf{A} \cdot \mathbf{I}) + \tfrac{1}{3}\,\mathbf{A}^T \mathbf{A}^T \cdot \mathbf{A} \\ &= \tfrac{1}{6}\,[(\operatorname{tr} \mathbf{A})^3 - 3 \operatorname{tr} \mathbf{A} \operatorname{tr}(\mathbf{A}\,\mathbf{A}) + 2 \operatorname{tr}(\mathbf{A}\,\mathbf{A}\,\mathbf{A})] = \det \mathbf{A} \quad (**) \end{aligned}$$

Note that (∗∗) yields the same result as (∗) in Subsection 2.8.

Eigenvalue representation:

$$
\begin{aligned}
I_{\mathbf{A}} &= \gamma_{\mathbf{A}(1)} + \gamma_{\mathbf{A}(2)} + \gamma_{\mathbf{A}(3)} \\
II_{\mathbf{A}} &= \gamma_{\mathbf{A}(1)}\,\gamma_{\mathbf{A}(2)} + \gamma_{\mathbf{A}(2)}\,\gamma_{\mathbf{A}(3)} + \gamma_{\mathbf{A}(3)}\,\gamma_{\mathbf{A}(1)} \\
III_{\mathbf{A}} &= \gamma_{\mathbf{A}(1)}\,\gamma_{\mathbf{A}(2)}\,\gamma_{\mathbf{A}(3)}
\end{aligned}
$$

CALEY-HAMILTON-Theorem:

$$
\mathbf{AAA} - I_{\mathbf{A}}\,\mathbf{AA} + II_{\mathbf{A}}\,\mathbf{A} - III_{\mathbf{A}}\,\mathbf{I} = \mathbf{0}
$$

3 Fundamentals of vector and tensor analysis

3.1 Introduction of functions

Notation: There

$$\text{exist} \left\{ \begin{array}{ll} \phi(\,\cdot\,) & : \text{scalar-valued functions} \\ \mathbf{v}(\,\cdot\,) & : \text{vector-valued functions} \\ \mathbf{T}(\,\cdot\,) & : \text{tensor-valued functions} \end{array} \right\} \text{ of } (\,\cdot\,) \left\{ \begin{array}{l} \text{scalar variables} \\ \text{vector variables} \\ \text{tensor variables} \end{array} \right.$$

Example: $\phi(\mathbf{A})$: scalar-valued tensor function

Notions:

- **Domain** of a function: set of all possible values of independent variable quantities (variables); usually continuous
- **Range** of a function: set of all possible values of dependent variable quantities: $\phi(\,\cdot\,)$; $\mathbf{v}(\,\cdot\,)$; $\mathbf{T}(\,\cdot\,)$

3.2 Functions of scalar variables

here: Vector- and tensor-valued functions of real scalar variables

(a) Vector-valued functions of a single variable

It exists:

$$\mathbf{u} = \mathbf{u}(\alpha) \quad \text{with} \quad \left\{ \begin{array}{ll} \mathbf{u} & : \text{unique vector-valued function,} \\ & \text{ range in the open domain } \mathcal{V}^3 \\ \alpha & : \text{real scalar variable} \end{array} \right.$$

Derivative of $\mathbf{u}(\alpha)$ with the differential quotient:

$$\mathbf{w}(\alpha) := \mathbf{u}'(\alpha) := \frac{\mathrm{d}\mathbf{u}(\alpha)}{\mathrm{d}\alpha}$$

Differential of $\mathbf{u}(\alpha)$:

$$\mathrm{d}\mathbf{u} = \mathbf{u}'(\alpha)\,\mathrm{d}\alpha$$

Introduction of higher derivatives and differentials:

$$\mathrm{d}^2\mathbf{u} = \mathrm{d}(\mathrm{d}\mathbf{u}) = \mathbf{u}''(\alpha)\,\mathrm{d}\alpha^2 = \frac{\mathrm{d}^2\mathbf{u}(\alpha)}{\mathrm{d}\alpha^2}\,\mathrm{d}\alpha^2 \quad \text{etc.}$$

(b) Vector-valued functions of several variables

It exists:

$$\mathbf{u} = \mathbf{u}(\alpha,\, \beta,\, \gamma,\, ...) \quad \text{with} \quad \{\alpha,\, \beta,\, \gamma,\, ...\} \;:\; \text{real scalar variables}$$

Partial derivative of $\mathbf{u}(\alpha,\, \beta,\, \gamma,\, ...)$:

$$\mathbf{w}_\alpha(\alpha,\, \beta,\, \gamma,\, ...) := \frac{\partial \mathbf{u}(\,\cdot\,)}{\partial \alpha} =: \mathbf{u},_\alpha$$

Total differential of $\mathbf{u}(\alpha,\, \beta,\, \gamma,\, ...)$:

$$\mathrm{d}\mathbf{u} = \mathbf{u},_\alpha\, \mathrm{d}\alpha + \mathbf{u},_\beta\, \mathrm{d}\beta + \mathbf{u},_\gamma\, \mathrm{d}\gamma + \,\cdots$$

Higher partial derivative (examples):

$$\mathbf{u},_{\alpha\alpha} = \frac{\partial^2 \mathbf{u}(\,\cdot\,)}{\partial \alpha^2}\;; \quad \mathbf{u},_{\gamma\beta} = \frac{\partial^2 \mathbf{u}(\,\cdot\,)}{\partial \gamma\, \partial \beta}$$

Remark: The order of partial derivatives is permutable.

(c) Tensor functions of a single or of several variables

Remark: Tensor-valued functions are treated analogously to the above procedure.

(d) Derivative of products of functions

Some rules:

$$\begin{aligned} (\mathbf{a} \otimes \mathbf{b})' &= \mathbf{a}' \otimes \mathbf{b} + \mathbf{a} \otimes \mathbf{b}' \\ (\mathbf{A}\,\mathbf{B})' &= \mathbf{A}'\,\mathbf{B} + \mathbf{A}\,\mathbf{B}' \\ (\mathbf{A}^{-1})' &= -\mathbf{A}^{-1}\,\mathbf{A}'\,\mathbf{A}^{-1} \end{aligned}$$

3.3 Functions of vector and tensor variables

(a) The gradient operator

Remark: Functions of the position (placement) vector are called **field functions**. Derivatives with respect to the position vector are called "gradient of a function".

Scalar-valued functions $\phi(\mathbf{x})$

$$\operatorname{grad} \phi(\mathbf{x}) := \frac{\mathrm{d}\phi(\mathbf{x})}{\mathrm{d}\mathbf{x}} =: \mathbf{w}(\mathbf{x}) \quad \longrightarrow \quad \text{result is a vector field}$$

or in basis notation

$$\operatorname{grad} \phi(\mathbf{x}) := \frac{\partial \phi(\mathbf{x})}{\partial x_i}\, \mathbf{e}_i =: \phi,_i\, \mathbf{e}_i$$

Vector-valued functions $\mathbf{v}(\mathbf{x})$

$$\operatorname{grad}\mathbf{v}(\mathbf{x}) := \frac{\mathrm{d}\mathbf{v}(\mathbf{x})}{\mathrm{d}\mathbf{x}} =: \mathbf{S}(\mathbf{x}) \quad \longrightarrow \quad \text{result is a tensor field}$$

or in basis notation

$$\operatorname{grad}\mathbf{v}(\mathbf{x}) := \frac{\partial v_i(\mathbf{x})}{\partial x_j}\,\mathbf{e}_i \otimes \mathbf{e}_j =: v_{i,j}\;\mathbf{e}_i \otimes \mathbf{e}_j$$

Tensor-valued functions $\mathbf{T}(\mathbf{x})$

$$\operatorname{grad}\mathbf{T}(\mathbf{x}) := \frac{\mathrm{d}\mathbf{T}(\mathbf{x})}{\mathrm{d}\mathbf{x}} =: \overset{3}{\mathbf{U}}(\mathbf{x}) \quad \longrightarrow \quad \text{result is a tensor field of 3-rd order}$$

or in basis notation

$$\operatorname{grad}\mathbf{T}(\mathbf{x}) := \frac{\partial t_{ik}(\mathbf{x})}{\partial x_j}\,\mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_j =: t_{ik,j}\;\mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_j$$

Remark: The gradient operator $\operatorname{grad}(\,\cdot\,) = \nabla(\,\cdot\,)$ (with ∇ : Nabla operator) increases the order of the respective function by one.

(b) Derivative of functions of arbitrary vectorial and tensorial variables

Remark: Derivatives concerning the respective variables are built analogously to the preceding procedures, e. g.

$$\frac{\partial \mathbf{R}(\mathbf{T},\,\mathbf{v})}{\partial \mathbf{T}} = \frac{\partial R_{ij}(\mathbf{T},\,\mathbf{v})}{\partial t_{st}}\,\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_s \otimes \mathbf{e}_t$$

Some specific rules for the derivative of tensor functions with respect to tensors

For arbitrary 2nd order tensors $\mathbf{A}, \mathbf{B}, \mathbf{C}$, the following rules hold:

$$\begin{aligned}
\frac{\partial \mathbf{A}}{\partial \mathbf{A}} &= (\mathbf{I}\otimes\mathbf{I})^{\overset{23}{T}} =: \overset{4}{\mathbf{I}} \\
\frac{\partial \mathbf{A}^T}{\partial \mathbf{A}} &= (\mathbf{I}\otimes\mathbf{I})^{\overset{24}{T}} \\
\frac{\partial (\mathbf{A}\cdot\mathbf{I})\,\mathbf{I}}{\partial \mathbf{A}} &= (\mathbf{I}\otimes\mathbf{I}) \\
\frac{\partial \overset{\mathrm{A}}{\mathbf{t}}(\mathbf{A})}{\partial \mathbf{A}} &= -\tfrac{1}{2}\;\overset{3}{\mathbf{E}} \\
\frac{\partial (\mathbf{A}\mathbf{B})}{\partial \mathbf{B}} &= (\mathbf{A}\otimes\mathbf{I})^{\overset{23}{T}} \\
\frac{\partial (\mathbf{A}\mathbf{B})}{\partial \mathbf{A}} &= (\mathbf{I}\otimes\mathbf{B}^T)^{\overset{23}{T}}
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial(\mathbf{A}\mathbf{A})}{\partial\mathbf{A}} &= (\mathbf{A}\otimes\mathbf{I})^{\overset{23}{T}} + (\mathbf{I}\otimes\mathbf{A}^T)^{\overset{23}{T}} \\
\frac{\partial(\mathbf{A}^T\mathbf{A})}{\partial\mathbf{A}} &= (\mathbf{A}^T\otimes\mathbf{I})^{\overset{23}{T}} + (\mathbf{I}\otimes\mathbf{A})^{\overset{24}{T}} \\
\frac{\partial(\mathbf{A}\mathbf{A}^T)}{\partial\mathbf{A}} &= (\mathbf{A}\otimes\mathbf{I})^{\overset{24}{T}} + (\mathbf{I}\otimes\mathbf{A})^{\overset{23}{T}} \\
\frac{\partial(\mathbf{A}^T\mathbf{A}^T)}{\partial\mathbf{A}} &= (\mathbf{I}\otimes\mathbf{A}^T)^{\overset{24}{T}} + (\mathbf{A}^T\otimes\mathbf{I})^{\overset{24}{T}} \\
\frac{\partial(\mathbf{A}\mathbf{B}\mathbf{C})}{\partial\mathbf{B}} &= (\mathbf{A}\otimes\mathbf{C}^T)^{\overset{23}{T}} \\
\frac{\partial\mathbf{A}^{-1}}{\partial\mathbf{A}} &= -(\mathbf{A}^{-1}\otimes\mathbf{A}^{T-1})^{\overset{23}{T}} \\
\frac{\partial\mathbf{A}^{T-1}}{\partial\mathbf{A}} &= -(\mathbf{A}^{T-1}\otimes\mathbf{A}^{T-1})^{\overset{24}{T}} \\
\frac{\partial\overset{+}{\mathbf{A}}}{\partial\mathbf{A}} &= \det\mathbf{A}\,[(\mathbf{A}^{T-1}\otimes\mathbf{A}^{T-1}) - (\mathbf{A}^{T-1}\otimes\mathbf{A}^{T-1})^{\overset{24}{T}}] \\
\frac{\partial(\alpha\,\beta)}{\partial\mathbf{C}} &= \alpha\,\frac{\partial\beta}{\partial\mathbf{C}} + \beta\,\frac{\partial\alpha}{\partial\mathbf{C}} \\
\frac{\partial(\alpha\,\mathbf{v})}{\partial\mathbf{C}} &= \mathbf{v}\otimes\frac{\partial\alpha}{\partial\mathbf{C}} + \alpha\,\frac{\partial\mathbf{v}}{\partial\mathbf{C}} \\
\frac{\partial(\alpha\,\mathbf{A})}{\partial\mathbf{C}} &= \mathbf{A}\otimes\frac{\partial\alpha}{\partial\mathbf{C}} + \alpha\,\frac{\partial\mathbf{A}}{\partial\mathbf{C}} \\
\frac{\partial(\mathbf{A}\,\mathbf{v})}{\partial\mathbf{C}} &= \left[\left(\frac{\partial\mathbf{A}}{\partial\mathbf{C}}\right)^{\overset{24}{T}}\right]^{\overset{23}{T}}\mathbf{v} + \left[\mathbf{A}\,\frac{\partial\mathbf{v}}{\partial\mathbf{C}}\right]_{\underline{3}} \\
\frac{\partial(\mathbf{u}\cdot\mathbf{v})}{\partial\mathbf{C}} &= \left[\left(\frac{\partial\mathbf{u}}{\partial\mathbf{C}}\right)^{\overset{13}{T}}\mathbf{v}\right]^T + \left[\left(\frac{\partial\mathbf{v}}{\partial\mathbf{C}}\right)^{\overset{13}{T}}\mathbf{u}\right]^T \\
\frac{\partial(\mathbf{A}\cdot\mathbf{B})}{\partial\mathbf{C}} &= \left(\frac{\partial\mathbf{A}}{\partial\mathbf{C}}\right)^T\mathbf{B} + \left(\frac{\partial\mathbf{B}}{\partial\mathbf{C}}\right)^T\mathbf{A} \\
\frac{\partial(\mathbf{A}\mathbf{B})}{\partial\mathbf{C}} &= \left(\left[\left(\frac{\partial\mathbf{A}}{\partial\mathbf{C}}\right)^{\overset{24}{T}}\mathbf{B}\right]_{\underline{4}}\right)^{\overset{24}{T}} + \left(\left[\left(\frac{\partial\mathbf{B}}{\partial\mathbf{C}}\right)^{\overset{14}{T}}\mathbf{A}^T\right]_{\underline{4}}\right)^{\overset{14}{T}}
\end{aligned}
$$

Principal invariants and their derivatives (see also section 2.9)

$$
\begin{aligned}
\frac{\partial I_{\mathbf{A}}}{\partial\mathbf{A}} &= \mathbf{I} && \text{with} && I_{\mathbf{A}} = \mathbf{A}\cdot\mathbf{I} \\
\frac{\partial II_{\mathbf{A}}}{\partial\mathbf{A}} &= \mathbf{A}\divideontimes\mathbf{I} && \text{with} && II_{\mathbf{A}} = \tfrac{1}{2}\,(I_{\mathbf{A}}^2 - \mathbf{A}\,\mathbf{A}\cdot\mathbf{I}) \\
\frac{\partial III_{\mathbf{A}}}{\partial\mathbf{A}} &= \overset{+}{\mathbf{A}} && \text{with} && III_{\mathbf{A}} = \det\mathbf{A}
\end{aligned}
$$

(c) Specific operators

here: Introduction of the further differential operators div (·) and rot (·).

Divergence of a vector field v(x)

$$\operatorname{div}\mathbf{v}(\mathbf{x}) := \operatorname{grad}\mathbf{v}(\mathbf{x})\cdot\mathbf{I} =: \phi(\mathbf{x}) \quad\longrightarrow\quad \text{result is a scalar field}$$

or in basis notation

$$\begin{aligned}\operatorname{div}\mathbf{v}(\mathbf{x}) &= v_{i},_{j}\ (\mathbf{e}_i\otimes\mathbf{e}_j)\cdot(\mathbf{e}_n\otimes\mathbf{e}_n)\\ &= v_{i},_{j}\ \delta_{in}\,\delta_{jn} = v_{n},_{n}\\ &= \frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}\end{aligned}$$

Divergence of a tensor field T(x)

$$\operatorname{div}\mathbf{T}(\mathbf{x}) = [\operatorname{grad}\mathbf{T}(\mathbf{x})]\,\mathbf{I} =: \mathbf{v}(\mathbf{x}) \quad\longrightarrow\quad \text{result is a vector field}$$

or in basis notation

$$\begin{aligned}\operatorname{div}\mathbf{T}(\mathbf{x}) &= t_{ik},_{j}\ (\mathbf{e}_i\otimes\mathbf{e}_k\otimes\mathbf{e}_j)\,(\mathbf{e}_n\otimes\mathbf{e}_n)\\ &= t_{ik},_{j}\ \delta_{kn}\,\delta_{jn}\,\mathbf{e}_i = t_{in},_{n}\ \mathbf{e}_i\end{aligned}$$

Remark: The divergence operator div (·) = ∇ · (·) decreases the order of the respective function by one.

Rotation of a vector field v(x)

$$\operatorname{rot}\mathbf{v}(\mathbf{x}) := \overset{3}{\mathbf{E}}\ [\operatorname{grad}\mathbf{v}(\mathbf{x})]^T =: \mathbf{r}(\mathbf{x}) \quad\longrightarrow\quad \text{result is a vector field}$$

or in basis notation

$$\begin{aligned}\operatorname{rot}\mathbf{v}(\mathbf{x}) &= \mathrm{e}_{ijn}\,(\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_n)\,v_{o},_{p}\ (\mathbf{e}_p\otimes\mathbf{e}_o)\\ &= \mathrm{e}_{ijn}\,v_{o},_{p}\ \delta_{jp}\,\delta_{no}\,\mathbf{e}_i = \mathrm{e}_{ijn}\,v_{n},_{j}\ \mathbf{e}_i\end{aligned}$$

Consequence: rot **v**(**x**) yields twice the axial vector corresponding to the skew-symmetric part of grad **v**(**x**).

Remark: The rotation operator rot (·) = curl (·) = ∇ × (·) preserves the order of the respective function.

Laplace **operator**

$$\Delta(\,\cdot\,) := \operatorname{div}\operatorname{grad}(\,\cdot\,) \ \longrightarrow\ \text{analogue to the previous}$$

Pierre-Simon Laplace (1749-1827), since 1817 Marquis de Laplace, was a French mathematician, physicist and astronomer.

Remark: The Laplace operator Δ(·) = ∇·∇(·) preserves the order of the differentiated function.

Rules for the operators grad (·), div (·), and rot (·)

$$
\begin{aligned}
\operatorname{grad}(\phi\psi) &= \phi \operatorname{grad}\psi + \psi \operatorname{grad}\phi \\
\operatorname{grad}(\phi\mathbf{v}) &= \mathbf{v}\otimes\operatorname{grad}\phi + \phi\operatorname{grad}\mathbf{v} \\
\operatorname{grad}(\phi\mathbf{T}) &= \mathbf{T}\otimes\operatorname{grad}\phi + \phi\operatorname{grad}\mathbf{T} \\
\operatorname{grad}(\mathbf{u}\cdot\mathbf{v}) &= (\operatorname{grad}\mathbf{u})^T\,\mathbf{v} + (\operatorname{grad}\mathbf{v})^T\,\mathbf{u} \\
\operatorname{grad}(\mathbf{u}\times\mathbf{v}) &= \mathbf{u}\times\operatorname{grad}\mathbf{v} + \operatorname{grad}\mathbf{u}\times\mathbf{v} \\
\operatorname{grad}(\mathbf{a}\otimes\mathbf{b}) &= [\operatorname{grad}\mathbf{a}\otimes\mathbf{b} + \mathbf{a}\otimes(\operatorname{grad}\mathbf{b})^T]^{\overset{23}{T}} \\
\operatorname{grad}(\mathbf{T}\mathbf{v}) &= (\operatorname{grad}\mathbf{T})^{\overset{23}{T}}\mathbf{v} + \mathbf{T}\operatorname{grad}\mathbf{v} \\
\operatorname{grad}(\mathbf{T}\mathbf{S}) &= [(\operatorname{grad}\mathbf{T})^{\overset{23}{T}}\mathbf{S}]^{\underline{3}\overset{23}{T}} + (\mathbf{T}\operatorname{grad}\mathbf{S})^{\underline{3}} \\
\operatorname{grad}(\overset{3}{\mathbf{T}}\mathbf{S}) &= (\operatorname{grad}\overset{3}{\mathbf{T}}{}^{\overset{23}{T}})^{\overset{24}{T}}\mathbf{S} + (\overset{3}{\mathbf{T}}\operatorname{grad}\mathbf{S})^{\underline{2}} \\
\operatorname{grad}(\mathbf{T}\cdot\mathbf{S}) &= (\operatorname{grad}\mathbf{T})^{\overset{13}{T}}\,\mathbf{S}^T + (\operatorname{grad}\mathbf{S})^{\overset{13}{T}}\,\mathbf{T}^T \\
\operatorname{grad}\mathbf{x} &= \mathbf{I} \\
\operatorname{div}(\mathbf{u}\otimes\mathbf{v}) &= \mathbf{u}\operatorname{div}\mathbf{v} + (\operatorname{grad}\mathbf{u})\,\mathbf{v} \\
\operatorname{div}(\phi\,\mathbf{v}) &= \mathbf{v}\cdot\operatorname{grad}\phi + \phi\operatorname{div}\mathbf{v} \\
\operatorname{div}(\mathbf{T}\mathbf{v}) &= (\operatorname{div}\mathbf{T}^T)\cdot\mathbf{v} + \mathbf{T}^T\cdot\operatorname{grad}\mathbf{v} \\
\operatorname{div}(\operatorname{grad}\mathbf{v})^T &= \operatorname{grad}\operatorname{div}\mathbf{v} \\
\operatorname{div}(\mathbf{u}\times\mathbf{v}) &= (\operatorname{grad}\mathbf{u}\times\mathbf{v})\cdot\mathbf{I} - (\operatorname{grad}\mathbf{v}\times\mathbf{u})\cdot\mathbf{I} \\
&= \mathbf{v}\cdot\operatorname{rot}\mathbf{u} - \mathbf{u}\cdot\operatorname{rot}\mathbf{v} \\
\operatorname{div}(\phi\,\mathbf{T}) &= \mathbf{T}\operatorname{grad}\phi + \phi\operatorname{div}\mathbf{T} \\
\operatorname{div}(\mathbf{T}\,\mathbf{S}) &= (\operatorname{grad}\mathbf{T})\,\mathbf{S} + \mathbf{T}\operatorname{div}\mathbf{S} \\
\operatorname{div}(\overset{3}{\mathbf{T}}\,\mathbf{S}) &= (\operatorname{div}\overset{3}{\mathbf{T}}{}^{\overset{13}{T}})\;\cdot\;\mathbf{S}^T + \;\overset{3}{\mathbf{T}}{}^{\overset{13}{T}}\;\cdot\;\operatorname{grad}\mathbf{S}^T \;\;(\dagger) \\
\operatorname{div}(\mathbf{v}\times\mathbf{T}) &= \mathbf{v}\times\operatorname{div}\mathbf{T} + \operatorname{grad}\mathbf{v}\times\mathbf{T} \\
\operatorname{div}(\mathbf{v}\otimes\mathbf{T}) &= \mathbf{v}\otimes\operatorname{div}\mathbf{T} + (\operatorname{grad}\mathbf{v})\,\mathbf{T}^T \\
\operatorname{div}(\mathbf{v}\otimes\overset{3}{\mathbf{T}}) &= \mathbf{v}\otimes\operatorname{div}\overset{3}{\mathbf{T}} + [(\operatorname{grad}\mathbf{v})\,(\,\overset{3}{\mathbf{T}}{}^{\overset{13}{T}})^{\overset{23}{T}}]^{\underline{3}} \\
\operatorname{div}(\operatorname{grad}\mathbf{v})^+ &= \mathbf{0} \\
\operatorname{div}[\operatorname{grad}\mathbf{v}\pm(\operatorname{grad}\mathbf{v})^T] &= \operatorname{div}\operatorname{grad}\mathbf{v}\pm\operatorname{grad}\operatorname{div}\mathbf{v} \\
\operatorname{div}\operatorname{rot}\mathbf{v} &= 0 \\
\operatorname{rot}\operatorname{rot}\mathbf{v} &= \operatorname{grad}\operatorname{div}\mathbf{v} - \operatorname{div}\operatorname{grad}\mathbf{v}
\end{aligned}
$$

$$\begin{aligned}
\operatorname{rot}\operatorname{grad}\mathbf{v} &= \mathbf{0} \\
\operatorname{rot}(\operatorname{grad}\mathbf{v})^T &= \operatorname{grad}\operatorname{rot}\mathbf{v} \\
\operatorname{rot}(\phi\,\mathbf{v}) &= \phi\operatorname{rot}\mathbf{v} + \operatorname{grad}\phi\times\mathbf{v} \\
\operatorname{rot}(\mathbf{u}\times\mathbf{v}) &= \operatorname{div}(\mathbf{u}\otimes\mathbf{v} - \mathbf{v}\otimes\mathbf{u}) \\
&= \mathbf{u}\operatorname{div}\mathbf{v} + (\operatorname{grad}\mathbf{u})\mathbf{v} - \mathbf{v}\operatorname{div}\mathbf{u} - (\operatorname{grad}\mathbf{v})\mathbf{u}
\end{aligned}$$

Grassmann **evolution:**

$$\mathbf{v}\times\operatorname{rot}\mathbf{v} = \tfrac{1}{2}\operatorname{grad}(\mathbf{v}\cdot\mathbf{v}) - (\operatorname{grad}\mathbf{v})\,\mathbf{v} = (\operatorname{grad}\mathbf{v})^T\mathbf{v} - (\operatorname{grad}\mathbf{v})\,\mathbf{v}$$

Hermann Günther Grassmann (1809-1877) was a German mathematician. He is one of the fathers of vector and tensor calculus.

3.4 Integral theorems

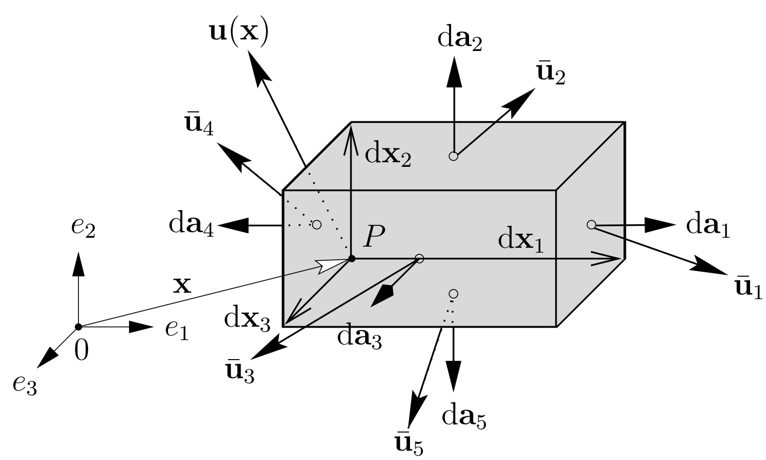
Remark: In what follows, some integral theorems for the transformation of surface integrals into volume integrals are presented.

Requirement: $\mathbf{u} = \mathbf{u}(\mathbf{x})$ is a steady and sufficiently often steadily differentiable vector field. The domain of $\mathbf{u}$ is in $\mathcal{V}^3$.

(a) Proof of the integral theorem

$$\int_{\mathcal{S}} \mathbf{u}(\mathbf{x})\otimes \mathrm{d}\mathbf{a} = \int_{V} \operatorname{grad}\mathbf{u}(\mathbf{x})\mathrm{d}v \qquad \text{with } \mathrm{d}\mathbf{a} = \mathbf{n}\,\mathrm{d}a$$

$$\text{and}\ \begin{cases} \mathrm{d}a & : \text{surface element} \\ \mathbf{n} & : \text{outward-oriented unit surface normal vector} \end{cases}$$



Basis: Consideration of an infinitesimal volume element $\mathrm{d}v$ spanned in the point P by the position vector $\mathbf{x}$, while $\bar{\mathbf{u}}_i$ defines the values of $\mathbf{u}(\mathbf{x})$ in the centroid of the partial surfaces 1 - 6.

Determination of the surface element vectors $\mathrm{d}\mathbf{a}_i$**:**

$$\begin{aligned}\mathrm{d}\mathbf{a}_1 &= \mathrm{d}\mathbf{x}_2 \times \mathrm{d}\mathbf{x}_3 = \mathrm{d}x_2\,\mathrm{d}x_3\,(\mathbf{e}_2 \times \mathbf{e}_3)\\ &= \mathrm{d}x_2\,\mathrm{d}x_3\,\mathbf{e}_1 = -\mathrm{d}\mathbf{a}_4 \;\longrightarrow\; \mathbf{e}_1 = \mathbf{n}_1 = -\mathbf{n}_4\end{aligned}$$

Furthermore, one obtains

$$\begin{aligned}\mathrm{d}\mathbf{a}_2 &= \mathrm{d}x_3\,\mathrm{d}x_1\,\mathbf{e}_2 = -\mathrm{d}\mathbf{a}_5 \;\longrightarrow\; \mathbf{e}_2 = \mathbf{n}_2 = -\mathbf{n}_5\\ \mathrm{d}\mathbf{a}_3 &= \mathrm{d}x_1\,\mathrm{d}x_2\,\mathbf{e}_3 = -\mathrm{d}\mathbf{a}_6 \;\longrightarrow\; \mathbf{e}_3 = \mathbf{n}_3 = -\mathbf{n}_6\end{aligned}$$

Remark: The surface vectors hold the condition $\displaystyle\sum_{i=1}^{6} \mathrm{d}\mathbf{a}_i = \mathbf{0}$.

Determination of the volume elements $\mathrm{d}v$**:**

$$\mathrm{d}v = (\mathrm{d}\mathbf{x}_1 \times \mathrm{d}\mathbf{x}_2) \cdot \mathrm{d}\mathbf{x}_3 = \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$$

Values of $\mathbf{u}(\mathbf{x})$ **in the centroids of the partial surfaces:**

Remark: The increments of $\mathbf{u}(\mathbf{x})$ in the directions of $\mathrm{d}x_1$, $\mathrm{d}x_2$, $\mathrm{d}x_3$ are approximated by the first term of a TAYLOR series.

$$\begin{aligned}\bar{\mathbf{u}}_4 &= \mathbf{u}(\mathbf{x}) + \frac{1}{2}\frac{\partial \mathbf{u}}{\partial x_2}\,\mathrm{d}x_2 + \frac{1}{2}\frac{\partial \mathbf{u}}{\partial x_3}\,\mathrm{d}x_3\\ \bar{\mathbf{u}}_1 &= \bar{\mathbf{u}}_4 + \frac{\partial \mathbf{u}}{\partial x_1}\,\mathrm{d}x_1\end{aligned}$$

Furthermore, one obtains

$$\bar{\mathbf{u}}_2 = \bar{\mathbf{u}}_5 + \frac{\partial \mathbf{u}}{\partial x_2}\,\mathrm{d}x_2\,, \quad \bar{\mathbf{u}}_3 = \bar{\mathbf{u}}_6 + \frac{\partial \mathbf{u}}{\partial x_3}\,\mathrm{d}x_3$$

Computation of the surface integral yields

$$\int\limits_{\mathcal{S}(\mathrm{d}v)} \mathbf{u}(\mathbf{x}) \otimes \mathrm{d}\mathbf{a} \;\longrightarrow\; \sum_{i=1}^{6} \bar{\mathbf{u}}_i \otimes \mathrm{d}\mathbf{a}_i = \bar{\mathbf{u}}_1 \otimes \mathrm{d}\mathbf{a}_1 + \underbrace{\bar{\mathbf{u}}_4 \otimes \mathrm{d}\mathbf{a}_4}_{(\bar{\mathbf{u}}_1 - \frac{\partial \mathbf{u}}{\partial x_1}\,\mathrm{d}x_1) \otimes (-\mathrm{d}\mathbf{a}_1)} + \cdots$$

Thus

$$\sum_{i=1}^{6} \bar{\mathbf{u}}_i \otimes \mathrm{d}\mathbf{a}_i = \frac{\partial \mathbf{u}}{\partial x_1}\,\mathrm{d}x_1 \otimes \mathrm{d}\mathbf{a}_1 + \frac{\partial \mathbf{u}}{\partial x_2}\,\mathrm{d}x_2 \otimes \mathrm{d}\mathbf{a}_2 + \frac{\partial \mathbf{u}}{\partial x_3}\,\mathrm{d}x_3 \otimes \mathrm{d}\mathbf{a}_3$$

with

$$\mathrm{d}\mathbf{a}_1 = \mathrm{d}x_2\,\mathrm{d}x_3\,\mathbf{e}_1\,,\; \mathrm{d}\mathbf{a}_2 = \mathrm{d}x_1\,\mathrm{d}x_3\,\mathbf{e}_2\,,\; \mathrm{d}\mathbf{a}_3 = \mathrm{d}x_1\,\mathrm{d}x_2\,\mathbf{e}_3$$

yields

$$\sum_{i=1}^{6} \bar{\mathbf{u}}_i \otimes \mathrm{d}\mathbf{a}_i = \underbrace{\left(\frac{\partial \mathbf{u}}{\partial x_1} \otimes \mathbf{e}_1 + \frac{\partial \mathbf{u}}{\partial x_2} \otimes \mathbf{e}_2 + \frac{\partial \mathbf{u}}{\partial x_3} \otimes \mathbf{e}_3\right)}_{\frac{\partial u_i}{\partial x_j}\,\mathbf{e}_i \otimes \mathbf{e}_j = \operatorname{grad}\mathbf{u}} \underbrace{\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3}_{\mathrm{d}v}$$

Thus

$$\sum_{i=1}^{6} \bar{\mathbf{u}}_i \otimes \mathrm{d}\mathbf{a}_i = \operatorname{grad} \mathbf{u} \, \mathrm{d}v$$

Integration over an arbitrary volume $\mathcal{V}$ yields

$$\int_{\mathcal{S}} \mathbf{u}(\mathbf{x}) \otimes \mathrm{d}\mathbf{a} = \int_{\mathcal{V}} \operatorname{grad} \mathbf{u}(\mathbf{x}) \, \mathrm{d}v \quad \text{q. e. d.} \qquad (*)$$

(b) Proof of the GAUSSian integral theorem

Johann Carl Friedrich Gauss (1777-1855) was a German mathematician, astronomer, geodesist, and physicist who contributed to many fields in mathematics and science.

$$\int_{\mathcal{S}} \mathbf{u}(\mathbf{x}) \cdot \mathrm{d}\mathbf{a} = \int_{\mathcal{V}} \operatorname{div} \mathbf{u}(\mathbf{x}) \, \mathrm{d}v$$

Basis: Integral theorem $(*)$ after scalar multiplication with the identity tensor

$$\begin{aligned} \mathbf{I} \cdot \int_{\mathcal{S}} \mathbf{u}(\mathbf{x}) \otimes \mathrm{d}\mathbf{a} \quad &= \mathbf{I} \cdot \int_{\mathcal{V}} \operatorname{grad} \mathbf{u}(\mathbf{x}) \, \mathrm{d}v \\ \longrightarrow \quad \int_{\mathcal{S}} \underbrace{\mathbf{I} \cdot [\mathbf{u}(\mathbf{x}) \otimes \mathrm{d}\mathbf{a}]}_{\mathbf{u}(\mathbf{x}) \cdot \mathrm{d}\mathbf{a}} \quad &= \int_{\mathcal{V}} \underbrace{\mathbf{I} \cdot \operatorname{grad} \mathbf{u}(\mathbf{x})}_{\operatorname{div} \mathbf{u}(\mathbf{x})} \, \mathrm{d}v \end{aligned}$$

Thus, leading to

$$\int_{\mathcal{S}} \mathbf{u}(\mathbf{x}) \cdot \mathrm{d}\mathbf{a} = \int_{\mathcal{V}} \operatorname{div} \mathbf{u}(\mathbf{x}) \, \mathrm{d}v \qquad (**)$$

(c) Proof of the integral theorem

$$\int_{\mathcal{S}} \mathbf{T}(\mathbf{x}) \, \mathrm{d}\mathbf{a} = \int_{\mathcal{V}} \operatorname{div} \mathbf{T}(\mathbf{x}) \, \mathrm{d}v$$

Basis: Scalar multiplication of the surface integral with a constant vector $\mathbf{b} \in \mathcal{V}^3$

$$\mathbf{b} \cdot \int_{\mathcal{S}} \mathbf{T}(\mathbf{x}) \, \mathrm{d}\mathbf{a} = \int_{\mathcal{S}} \mathbf{b} \cdot \mathbf{T}(\mathbf{x}) \, \mathrm{d}\mathbf{a} = \int_{\mathcal{S}} [\mathbf{T}^T(\mathbf{x}) \, \mathbf{b}] \cdot \mathrm{d}\mathbf{a} =: \int_{\mathcal{S}} \mathbf{u}(\mathbf{x}) \cdot \mathrm{d}\mathbf{a}$$

with $\quad \mathbf{u}(\mathbf{x}) := \mathbf{T}^T(\mathbf{x}) \, \mathbf{b}$

It follows with the integral theorem $(**)$

$$\mathbf{b} \cdot \int_{\mathcal{S}} \mathbf{T}(\mathbf{x}) \, \mathrm{d}\mathbf{a} = \int_{\mathcal{V}} \operatorname{div} [\mathbf{T}^T(\mathbf{x}) \, \mathbf{b}] \, \mathrm{d}v$$

In particular, with $\mathbf{b} =$ const. and a divergence rule follows

$$\operatorname{div}\,[\mathbf{T}^T(\mathbf{x})\,\mathbf{b}] = \operatorname{div}\mathbf{T}(\mathbf{x})\cdot\mathbf{b}$$

leading to

$$\mathbf{b}\cdot\int\limits_{\mathcal{S}}\mathbf{T}(\mathbf{x})\,\mathrm{d}\mathbf{a} = \int\limits_{\mathcal{V}}\operatorname{div}\mathbf{T}(\mathbf{x})\cdot\mathbf{b}\,\mathrm{d}v$$

Thus

$$\int\limits_{\mathcal{S}}\mathbf{T}(\mathbf{x})\,\mathrm{d}\mathbf{a} = \int\limits_{\mathcal{V}}\operatorname{div}\mathbf{T}(\mathbf{x})\,\mathrm{d}v \quad \text{q. e. d.}$$

Remark: At this point, no further proofs are carried out.

(d) Summary of some integral theorems

For the transformation of surface into volume integrals, the following relations hold:

$$\begin{aligned}
\int\limits_{\mathcal{S}}\mathbf{u}\otimes\mathrm{d}\mathbf{a} &= \int\limits_{\mathcal{V}}\operatorname{grad}\mathbf{u}\,\mathrm{d}v\\
\int\limits_{\mathcal{S}}\phi\,\mathrm{d}\mathbf{a} &= \int\limits_{\mathcal{V}}\operatorname{grad}\phi\,\mathrm{d}v\\
\int\limits_{\mathcal{S}}\mathbf{u}\cdot\mathrm{d}\mathbf{a} &= \int\limits_{\mathcal{V}}\operatorname{div}\mathbf{u}\,\mathrm{d}v\ \ (*)\\
\int\limits_{\mathcal{S}}(\overset{3}{\mathbf{T}}{}^T\overset{13}{\mathbf{S}})\,\cdot\,\mathrm{d}\mathbf{a} &= \int\limits_{\mathcal{V}}\operatorname{div}(\overset{3}{\mathbf{T}}{}^T\overset{13}{\mathbf{S}})\,\mathrm{d}v = \int\limits_{\mathcal{V}}[\,(\operatorname{div}\overset{3}{\mathbf{T}})\,\mathbf{L}^T + \overset{3}{\mathbf{T}}\,\cdot\,\operatorname{grad}\mathbf{L}^T\,]\,\mathrm{d}v\\
\int\limits_{\mathcal{S}}\mathbf{u}\times\mathrm{d}\mathbf{a} &= -\int\limits_{\mathcal{V}}\operatorname{rot}\mathbf{u}\,\mathrm{d}v\\
\int\limits_{\mathcal{S}}\mathbf{T}\,\mathrm{d}\mathbf{a} &= \int\limits_{\mathcal{V}}\operatorname{div}\mathbf{T}\,\mathrm{d}v\\
\int\limits_{\mathcal{S}}\mathbf{u}\times\mathbf{T}\,\mathrm{d}\mathbf{a} &= \int\limits_{\mathcal{V}}\operatorname{div}(\mathbf{u}\times\mathbf{T})\,\mathrm{d}v\\
\int\limits_{\mathcal{S}}\mathbf{u}\otimes\mathbf{T}\,\mathrm{d}\mathbf{a} &= \int\limits_{\mathcal{V}}\operatorname{div}(\mathbf{u}\otimes\mathbf{T})\,\mathrm{d}v
\end{aligned}$$

Application example: The integral theorem $(*)$, as well as other theorems, can also be applied to complexer terms, such as $\mathbf{u} := (\overset{3}{\mathbf{T}}{}^{\overset{13}{T}}\mathbf{S})$. This yields by use of $(\dagger)$ from Subsection 3.3 (c)

$$\int_{\mathcal{S}} (\overset{3}{\mathbf{T}}{}^{\overset{13}{T}}\mathbf{S}) \cdot \mathrm{d}\mathbf{a} = \int_{\mathcal{V}} \operatorname{div} (\overset{3}{\mathbf{T}}{}^{\overset{13}{T}}\mathbf{S}) \mathrm{d}v = \int_{\mathcal{V}} [\,(\operatorname{div} \overset{3}{\mathbf{T}})\, \mathbf{S}^T + \overset{3}{\mathbf{T}} \cdot \operatorname{grad} \mathbf{S}^T\,] \mathrm{d}v$$

For the transformation of line into surface integrals, the following relations hold:

$$\begin{aligned}
\oint_{\mathcal{L}} \mathbf{u} \otimes \mathrm{d}\mathbf{x} &= -\int_{\mathcal{S}} \operatorname{grad} \mathbf{x} \times \mathrm{d}\mathbf{a} \\
\oint_{\mathcal{L}} \phi \, \mathrm{d}\mathbf{x} &= -\int_{\mathcal{S}} \operatorname{grad} \phi \times \mathrm{d}\mathbf{a} \\
\oint_{\mathcal{L}} \mathbf{u} \cdot \mathrm{d}\mathbf{x} &= \int_{\mathcal{S}} (\operatorname{rot} \mathbf{u}) \cdot \mathrm{d}\mathbf{a} \\
\oint_{\mathcal{L}} \mathbf{u} \times \mathrm{d}\mathbf{x} &= \int_{\mathcal{S}} (\mathbf{I} \operatorname{div} \mathbf{u} - \operatorname{grad}^T \mathbf{u})\, \mathrm{d}\mathbf{a} \\
\oint_{\mathcal{L}} \mathbf{T}\, \mathrm{d}\mathbf{x} &= \int_{\mathcal{S}} (\operatorname{rot} \mathbf{T})^T \mathrm{d}\mathbf{a}
\end{aligned}$$

3.5 Transformations between current and reference configurations

Given are the deformation gradient $\mathbf{F} = \partial \mathbf{x} / \partial \mathbf{X}$ and arbitrary vectorial and tensorial field functions $\mathbf{v}$ and $\mathbf{A}$. Then, with t_0 (reference) and t (current time)

$$\text{reference configuration} \quad \begin{cases} \operatorname{Grad}(\,\cdot\,) = \dfrac{\partial}{\partial \mathbf{X}}(\,\cdot\,) \\ \operatorname{Div}(\,\cdot\,) = [\operatorname{Grad}(\,\cdot\,)] \cdot \mathbf{I} \quad \text{or} \quad [\operatorname{Grad}(\,\cdot\,)]\,\mathbf{I} \end{cases}$$

$$\text{current configuration} \quad \begin{cases} \operatorname{grad}(\,\cdot\,) = \dfrac{\partial}{\partial \mathbf{x}}(\,\cdot\,) \\ \operatorname{div}(\,\cdot\,) = [\operatorname{grad}(\,\cdot\,)] \cdot \mathbf{I} \quad \text{or} \quad [\operatorname{grad}(\,\cdot\,)]\,\mathbf{I} \end{cases}$$

The following relations hold:

$$\begin{aligned}
\operatorname{Grad} \mathbf{v} &= (\operatorname{grad} \mathbf{v})\, \mathbf{F} & \qquad \operatorname{Grad} \mathbf{A} &= [(\operatorname{grad} \mathbf{A})\, \mathbf{F}]^{\underline{3}} \\
\operatorname{grad} \mathbf{v} &= (\operatorname{Grad} \mathbf{v})\, \mathbf{F}^{-1} & \qquad \operatorname{grad} \mathbf{A} &= [(\operatorname{Grad} \mathbf{A})\, \mathbf{F}^{-1}]^{\underline{3}} \\
\\
\operatorname{Div} \mathbf{v} &= (\operatorname{grad} \mathbf{v}) \cdot \mathbf{F}^T & \qquad \operatorname{Div} \mathbf{A} &= (\operatorname{grad} \mathbf{A})\, \mathbf{F}^T \\
\operatorname{div} \mathbf{v} &= (\operatorname{Grad} \mathbf{v}) \cdot \mathbf{F}^{T-1} & \qquad \operatorname{div} \mathbf{A} &= (\operatorname{Grad} \mathbf{A})\, \mathbf{F}^{T-1}
\end{aligned}$$

PIOLA identities ($\mathbf{T}$, $\mathbf{P}$, $\mathbf{S}$: Cauchy and 1st and 2nd Piola-Kirchhoff stress tensors)

$$
\begin{array}{llllll}
(1) & \mathrm{Div}\,\mathbf{v}_{0t} = (\det\mathbf{F})\,\mathrm{div}\,\mathbf{v} & (2) & \mathbf{v}_{0t} = (\det\mathbf{F})\,\mathbf{F}^{-1}\mathbf{v} & (3) & \mathrm{Div}\,(\mathrm{cof}\,\mathbf{F}) \equiv 0 \\
(4) & \mathrm{Div}\,\mathbf{P} = (\det\mathbf{F})\,\mathrm{div}\,\mathbf{T} & (5) & \mathbf{P} = (\det\mathbf{F})\mathbf{T}\,\mathbf{F}^{T-1} & (6) & \mathbf{S} = (\det\mathbf{F})\mathbf{F}^{-1}\mathbf{T}\,\mathbf{F}^{T-1}
\end{array}
$$

Therein, $\mathbf{v} = \mathbf{v}(t)$ is an arbitary vector acting at the current configuration at time t, such as the velocity. Then, $\mathbf{v}_0$ is its initial value at t_0, while $\mathbf{v}_{0t}$ represents the image of $\mathbf{v}(t)$ at the refence configuration. The same is true for the Cauchy stress $\mathbf{T}$ at the current configuration and the Piola stress $\mathbf{P}$ acting as the image of $\mathbf{T}$ at the reference configuration.

Furthermore, it can be shown that

$$
\begin{array}{lllll}
\mathrm{Div}\,\mathbf{F}^{T-1} & = & -\mathbf{F}^{T-1}\,(\mathbf{F}^{T-1}\,\mathrm{Grad}\,\mathbf{F})\overset{1}{\_} & = & -(\det\mathbf{F})^{-1}\,\mathbf{F}^{T-1}\,[\mathrm{Grad}\,(\det\mathbf{F})] \\
\mathrm{div}\,\mathbf{F}^{T} & = & -\mathbf{F}^{T}\,(\mathbf{F}^{T}\,\mathrm{grad}\,\mathbf{F}^{-1})\overset{1}{\_} & = & -(\det\mathbf{F})\,\mathbf{F}^{T}\,[\mathrm{grad}\,(\det\mathbf{F})^{-1}]
\end{array}
$$

Remark: If required, further relations of vector and tensor calculus can be constructed in the respective context.

The description of non-orthogonal and non-unit basis systems, such as a general or the natural basis, has not been discussed in this contribution so far. The interested reader can find this material in the Appendix to this Treatise.

However, as the complete material has been presented in an absolute vector and tensor notation, the entire material is independent of the choice of specific basis systems.

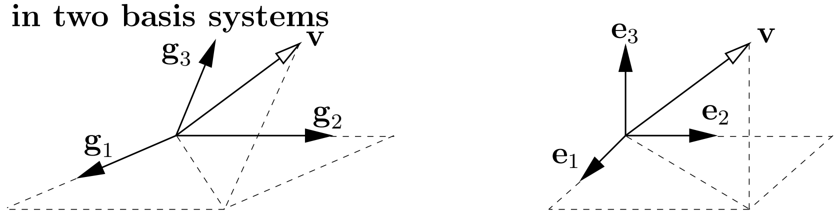
Appendix on Natural Basis Systems

A Differential geometry of continua

A.1 Tangent space and natural basis

In Section 1.1 B of the above vector and tensor treatise, an arbitrary vector $\mathbf{v} \in \mathcal{V}^3$ has been displayed both in a general basis system with non-coplanar basis vectors $\mathbf{g}_i$ of arbitrary lengths and in an orthonormal basis system $\mathbf{e}_i$, where the basis vectors $\mathbf{e}_i$ are perpendicular to each other with unit lengths, such that $|\mathbf{e}_i| = 1$.

Vector v in two basis systems

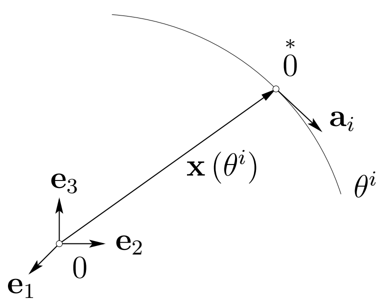


basis system $\mathbf{g}_i$ basis system $\mathbf{e}_i$

Representations of $\mathbf{v}$:
$$\begin{cases} \mathbf{v} \;=\; v_i\,\mathbf{e}_i \;=\; v_1\,\mathbf{e}_1 + v_2\,\mathbf{e}_2 + v_3\,\mathbf{e}_3 \\ \mathbf{v} \;=\; \bar{v}_i\,\mathbf{g}_i \;=\; \bar{v}_1\,\mathbf{g}_1 + \bar{v}_2\,\mathbf{g}_2 + \bar{v}_3\,\mathbf{g}_3 \end{cases}$$

Natural basis / covariant tangent vectors

In $\mathcal{V}^3$, tangent vectors $\mathbf{a}_i$ are introduced tangentially to the trajectories (curved parameter lines θ^i with $\theta^i \in \{\, \theta^1,\, \theta^2,\, \theta^3 \,\}$).



$\{\, 0,\, \mathbf{e}_i \,\}$: spatially fixed reference systen

$\{\, \overset{*}{0},\, \mathbf{a}_i \,\}$: tangent system

The tangent vectors are the **covariant basis vectors** defined as

$$\boxed{\mathbf{a}_i := \frac{\partial \mathbf{x}(\theta^i)}{\partial \theta^i} =: \mathbf{x}_{,i}}$$

A scalar multiplication of two tangent vectors $\mathbf{a}_i$ and $\mathbf{a}_k$ yields the "covariant" metric coefficients a_{ik}. The metric coefficients a_{ik} are symmetric, such that

$$a_{ik} \;=\; \mathbf{a}_i \cdot \mathbf{a}_k \;=\; \mathbf{a}_k \cdot \mathbf{a}_i \quad \longrightarrow \quad a_{ik} \;=\; a_{ki}$$

Note: As a result of the symmetry of a_{ik}, only 6 of the 9 entries of a_{ik} are independent quantities.

From the definition of the scalar product, the metric coefficients and the cosines of the angles between the $\mathbf{a}_i$ and $\mathbf{a}_k$ are connected to each other via

$$\mathbf{a}_i \cdot \mathbf{a}_k \;=\; |\mathbf{a}_i|\,|\mathbf{a}_k|\,\cos\,\sphericalangle(\mathbf{a}_i\,;\,\mathbf{a}_k) \quad \text{with} \quad |\mathbf{a}_i| = \sqrt{a_{ii}}$$

such that the angle $\sphericalangle(\mathbf{a}_i\,;\,\mathbf{a}_k) = \varphi_{ik}$ between $\mathbf{a}_i$ and $\mathbf{a}_k$ can be obtained as

$$\cos\varphi_{ik} = \frac{a_{ik}}{\sqrt{a_{ii}}\sqrt{a_{kk}}}$$

In the above computation of $\sqrt{a_{ii}}$ and $\sqrt{a_{kk}}$, there is no summation over i and k, also see the comments further below on this page.

Dual basis / contravariant cotangent vectors

Introduction of the dual basis via the property

$$\boxed{\mathbf{a}_i \cdot \mathbf{a}^k \;:=\; \delta_i^k = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \;\; \text{meaning that } \mathbf{a}_i \perp \mathbf{a}^k \end{cases}} \qquad (*)$$

Note: In case of $i = k$, $\mathbf{a}_i$ is only parallel to $\mathbf{a}^k$, if $|\mathbf{a}_i| = |\mathbf{a}^k| = 1$.

Dual metric coefficients

$$a_{ik}\,a^{kj} = \delta_i^j \quad\longrightarrow\quad \text{6 equations for 6 unknown entries in } a^{kj}$$

Computation of the dual basis

$$\mathbf{a}^k = a^{kj}\,\mathbf{a}_j$$

Validity control by use of $(*)$

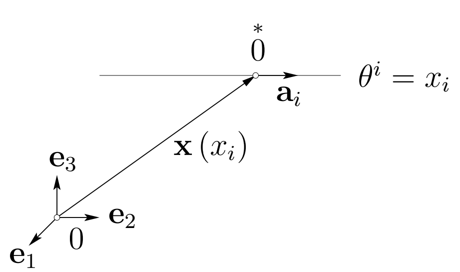
$$\mathbf{a}_i \cdot \mathbf{a}^k = \mathbf{a}_i \cdot a^{kj}\mathbf{a}_j = a_{ij}\,a^{kj} = \delta_i^k$$

Comments:

- $\mathbf{a}_i$ and a_{ij} define the tangent space (covariant)
- $\mathbf{a}^i$ and a^{ij} define the cotangent space (contravariant)
- tangent and cotangent vectors are via $(*)$ dual (inverse) to each other
- EINSTEIN's summation convention can only be applied, when a double index appears in opposite positions (co- and contravariant)

The special case of an orthonormal basis

For the special case of an orthonormal basis $\mathbf{e}_i$ with $|\mathbf{e}_i| = 1$, the general parameter lines θ^i are equal to the straight parameter lines x^i. Thus,



$$\mathbf{a}_i = \frac{\partial \mathbf{x}(x^i)}{\partial x^i} = \mathbf{e}_i$$

$$a_{ik} = \mathbf{a}_i \cdot \mathbf{a}_k = \mathbf{e}_i \cdot \mathbf{e}_k = \delta_{ik}$$

$$a_{ik}\,a^{kj} = \delta_i^j \longrightarrow a^{kj} = \delta^{kj}$$

$$\mathbf{a}^k = a^{kj}\,\mathbf{a}_j = \delta^{kj}\,\mathbf{e}_j = \mathbf{e}^k = \mathbf{e}_k \quad \text{and} \quad x^i = x_i$$

Notes:

- It is easily concluded that the basis vectors $\mathbf{e}_i$ and $\mathbf{e}_k$ are perpendicular to each other as far as $i \neq k$.
- In case of an orthonormal basis, tangent and cotangent space coincide yielding

$$\boxed{\mathbf{e}^i = \mathbf{e}_i}$$

A.2 Vector and tensor algebra in natural basis systems

Scalar or dot product

Note: The scalar or dot product between two objects, such as $\mathbf{u}$ and $\mathbf{v}$, is also called "inner product" as its result, a scalar, remains in terms of its metric coefficients in the same tangent or cotangent space as the objects have been before.

Given two vectors $\mathbf{u}$ and $\mathbf{v}$, for example, in a tangent and a cotangent basis system, the scalar product can be computed in two ways

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= |\mathbf{u}|\,|\mathbf{v}| \cos \sphericalangle(\mathbf{u}\,;\,\mathbf{v}) \\
&= \sqrt{(u^i \mathbf{a}_i) \cdot (u^k \mathbf{a}_k)} \sqrt{(v_n \mathbf{a}^n) \cdot (v_o \mathbf{a}^o)} \cos \sphericalangle(\mathbf{u}\,;\,\mathbf{v}) \\
&= \sqrt{u^i u^k a_{ik}} \sqrt{v_n v_o a^{no}} \cos \sphericalangle(\mathbf{u}\,;\,\mathbf{v}) = \sqrt{u^i u_i} \sqrt{v^n v_n} \cos \sphericalangle(\mathbf{u}\,;\,\mathbf{v})
\end{aligned}$$

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= (u^i\, \mathbf{a}_i) \cdot (v_k\, \mathbf{a}^k) = u^i\, v_k\, (\mathbf{a}_i \cdot \mathbf{a}^k) \\
&= u^i\, v_k\, \delta_i^k = u^i\, v_i = u^1\, v_1 + u^2\, v_2 + u^3\, v_3 \\
&= u_i\, v_k\, a^{ik} = u^i\, v^k\, a_{ik}
\end{aligned}$$

From both equations, the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$ can be computed as

$$\cos \sphericalangle(\mathbf{u}\,;\,\mathbf{v}) = \frac{\mathbf{u}\,\cdot\,\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \frac{u^i\, v_i}{\sqrt{u^j u_j} \sqrt{v^n v_n}}$$

Physical coefficients of vector components

The coefficients v_i of a vector $\mathbf{v} = v_i\, \mathbf{e}_i$ describe the real value of the vector components $v_i\, \mathbf{e}_i$ with $i = 1, 2, 3$. Coefficients with this property are called physical coefficients. In general, this is only true in orthonormal basis systems $\mathbf{e}_i$ where $|\mathbf{e}_i| = 1$. In case of natural basis systems with co- and contravariant basis vectors given through $\mathbf{a}_i$ and $\mathbf{a}^i$, this is generally not the case meaning that both $|\mathbf{a}_i|$ and $|\mathbf{a}^i|$ are $\neq 1$.

Thus, physical coefficients are introduced via

$$\mathbf{v} = v^i\, \mathbf{a}_i = |\mathbf{a}_i|\, v^i\, \frac{\mathbf{a}_i}{|\mathbf{a}_i|} = \sqrt{a_{ii}}\, v^i\, \frac{\mathbf{a}_i}{\sqrt{a_{ii}}} = \overset{*}{v}{}^i\, \overset{*}{\mathbf{a}}_i$$

with no summation over $(\,\cdot\,)_{ii}$. From the above, physical coefficients $\overset{*}{v}{}^i$ are defined by

$$\overset{*}{v}{}^i = |\mathbf{a}_i|\, v^i \quad \text{and} \quad |\overset{*}{\mathbf{a}}_i = 1|$$

Identity tensor of 2nd **order**

Note: The identity $\mathbf{I}$ is the fundamental tensor of 2nd order. Fundamental tensors are constructed only by basis vectors

$$\mathbf{I} = \operatorname{grad}\mathbf{x} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\mathbf{x}} = \frac{\partial\mathbf{x}}{\partial\theta^i} \otimes \frac{\partial\theta^i}{\partial\mathbf{x}} = \underbrace{\mathbf{x}_{,i}}_{\mathbf{a}_i} \otimes \underbrace{\operatorname{grad}\theta^i}_{\mathbf{a}^i} = \mathbf{a}_i \otimes \mathbf{a}^i$$

Control of the identical mapping using $\mathbf{v} = v^n\,\mathbf{a}_n$

$$\mathbf{v} \stackrel{?}{=} \mathbf{I}\,\mathbf{v} = (\mathbf{a}_i \otimes \mathbf{a}^i)\, v^n\,\mathbf{a}_n = v^n\,\delta^i_n\,\mathbf{a}_i = v^n\,\mathbf{a}_n \quad \text{q. e. d.}$$

Representations of the identity tensor

Pulling the indices of $\mathbf{I}$ up and down by use of metric coefficients results in

$$\boxed{\mathbf{I} = \mathbf{a}_i \otimes \mathbf{a}^i = \mathbf{a}^i \otimes \mathbf{a}_i = a^{ik}\mathbf{a}_i \otimes \mathbf{a}_k = a_{ik}\mathbf{a}^i \otimes \mathbf{a}^k}$$

Note: From the cotangent basis defined as $\mathbf{a}^i = \partial\theta^i/\partial\mathbf{x}$, one recognises that the dual or cotangential basis is considered inverse to $\mathbf{a}_i = \partial\mathbf{x}/\partial\theta^i$.

Furthermore, the second-order identity tensor $\mathbf{I}$ can be given in an arbitrary basis represented by $\mathbf{g}_i$ and $\mathbf{g}^k$ without changing its property, the identical map.

Transformation of basis systems

Consider a covariant natural basis system $\mathbf{a}_i$ that should be given as a function of an arbitrary covariant basis system $\mathbf{g}_i$. Then, the transformation tensor $\mathbf{T}$ can be constructed as follows:

$$\begin{aligned}
\mathbf{a}_i &= \mathbf{I}\,\mathbf{a}_i \quad \text{with} \quad \mathbf{I} = (\mathbf{g}_j \otimes \mathbf{g}^j) \quad \text{and} \quad \delta^k_i = \mathbf{g}_i \cdot \mathbf{g}^k \\
&= (\mathbf{g}_j \otimes \mathbf{g}^j)\,\mathbf{a}_i = (\mathbf{g}^j \cdot \mathbf{a}_i)\,\mathbf{g}_j = (\mathbf{g}^j \cdot \delta^k_i\mathbf{a}_k)\,\mathbf{g}_j \\
&= (\mathbf{g}^j \cdot \mathbf{a}_k)\,(\mathbf{g}_i \cdot \mathbf{g}^k)\,\mathbf{g}_j \\
&= (\mathbf{g}^j \cdot \mathbf{a}_k)\,(\mathbf{g}_j \otimes \mathbf{g}^k)\,\mathbf{g}_i =: \mathbf{T}\,\mathbf{g}_i
\end{aligned}$$

$$\text{with} \qquad \boxed{\mathbf{T} = (\mathbf{g}^j \cdot \mathbf{a}_k)\,(\mathbf{g}_j \otimes \mathbf{g}^k)}$$

Comments:

- As $\mathbf{T}$ does not only include $\mathbf{g}_j$ and $\mathbf{g}^k$ but also $\mathbf{a}_i$, both systems, $\mathbf{a}_i$ and $\mathbf{g}_i$, must be known in advance.
- From $\mathbf{a}_i = \mathbf{T}\,\mathbf{g}_i$, one concludes to $\mathbf{g}_i = \mathbf{T}^{-1}\,\mathbf{a}_i$. As $\mathbf{T}$ displays $\mathbf{a}_i$ as a function of $\mathbf{g}_i$ in the sense of a unique transformation, $\det\mathbf{T}$ is non-zero, such that $\mathbf{T}^{-1}$ exists.

By use of the same procedure as before, one obtains

$$\boxed{\mathbf{T}^{-1} = (\mathbf{a}^j \cdot \mathbf{g}_k)\,(\mathbf{a}_j \otimes \mathbf{a}^k)}$$

such that

$$\begin{aligned} \mathbf{T}^{-1}\,\mathbf{a}_i &= (\mathbf{a}^j \cdot \mathbf{g}_k)\,(\mathbf{a}_j \otimes \mathbf{a}^k)\,\mathbf{a}_i = (\mathbf{a}^j \cdot \mathbf{g}_k)\,(\mathbf{a}_i \cdot \mathbf{a}^k)\,\mathbf{a}_j \\ &= (\mathbf{a}^j \cdot \delta_i^k\,\mathbf{g}_k)\;\mathbf{a}_j = (\mathbf{a}_j \otimes \mathbf{a}^j)\;\mathbf{g}_i = \mathbf{I}\;\mathbf{g}_i = \mathbf{g}_i \qquad \text{q. e. d.} \end{aligned}$$

From the above transformations to find $\mathbf{T}$ and $\mathbf{T}^{-1}$, the following relations hold:

$$\begin{aligned} \mathbf{a}_i &= (\mathbf{a}_i \cdot \mathbf{g}^j)\,\mathbf{g}_j & \mathbf{g}_i &= (\mathbf{g}_i \cdot \mathbf{a}^j)\,\mathbf{a}_j \\ \mathbf{a}^i &= (\mathbf{a}^i \cdot \mathbf{g}_j)\,\mathbf{g}^j & \mathbf{g}^i &= (\mathbf{g}^i \cdot \mathbf{a}_j)\,\mathbf{a}^j \end{aligned}$$

This leads to the description of a vector $\mathbf{v}$ in two basis systems, $\mathbf{a}_i$ and $\mathbf{g}_i$:

$$\mathbf{v} = v^i_{(a)}\,\mathbf{a}_i = v^i_{(g)}\,\mathbf{g}_i$$

with the coefficients of the vector components reading

$$\begin{aligned} v^j_{(g)} &= v^i_{(a)}\,(\mathbf{a}_i \cdot \mathbf{g}^j) & v^j_{(a)} &= v^i_{(g)}\,(\mathbf{g}_i \cdot \mathbf{a}^j) \\ v_{j(g)} &= v_{i(a)}\,(\mathbf{a}^i \cdot \mathbf{g}_j) & v_{j(a)} &= v_{i(g)}\,(\mathbf{g}^i \cdot \mathbf{a}_j) \end{aligned}$$

With the above equations, the validity of $\mathbf{T}\mathbf{T}^{-1} = \mathbf{I}$ can be proven:

$$\begin{aligned} \mathbf{T}\,\mathbf{T}^{-1} &= [(\mathbf{g}^j \cdot \mathbf{a}_k)\,(\mathbf{g}_j \otimes \mathbf{g}^k)][(\mathbf{a}^n \cdot \mathbf{g}_o)\,(\mathbf{a}_n \otimes \mathbf{a}^o)] \\ &= (\mathbf{g}^j \cdot \mathbf{a}_k)(\mathbf{a}^n \cdot \mathbf{g}_o)(\mathbf{g}_j \otimes \mathbf{g}^k)(\mathbf{a}_n \otimes \mathbf{a}^o) \\ &= (\mathbf{g}^j \cdot \mathbf{a}_k)(\mathbf{a}^n \cdot \mathbf{g}_o)(\mathbf{g}^k \cdot \mathbf{a}_n)(\mathbf{g}_j \otimes \mathbf{a}^o) \end{aligned}$$

With $\mathbf{a}^o = (\mathbf{a}^o \cdot \mathbf{g}_p)\,\mathbf{g}^p$, it follows that

$$\begin{aligned} \mathbf{T}\,\mathbf{T}^{-1} &= (\mathbf{g}^j \cdot \mathbf{a}_k)\,\underbrace{(\mathbf{a}^n \cdot \mathbf{g}_o)(\mathbf{g}^k \cdot \mathbf{a}_n)}_{(*)}(\mathbf{a}^o \cdot \mathbf{g}_p)(\mathbf{g}_j \otimes \mathbf{g}^p) \\ \text{where } (*) &= (\mathbf{g}_o \otimes \mathbf{g}^k) \cdot (\mathbf{a}^n \otimes \mathbf{a}_n) = (\mathbf{g}_o \otimes \mathbf{g}^k) \cdot \mathbf{I} \\ &= (\mathbf{g}_o \otimes \mathbf{g}^k) \cdot (\mathbf{g}^n \otimes \mathbf{g}_n) = \delta_o^n\,\delta_n^k = \delta_o^k \\ \text{such that} \quad \mathbf{T}\mathbf{T}^{-1} &= (\mathbf{g}^j \cdot \mathbf{a}_k)\,\delta_o^k\,(\mathbf{a}^o \cdot \mathbf{g}_p)(\mathbf{g}_j \otimes \mathbf{g}^p) \\ &= \underbrace{(\mathbf{g}^j \cdot \mathbf{a}_k)(\mathbf{a}^k \cdot \mathbf{g}_p)}_{(**)}(\mathbf{g}_j \otimes \mathbf{g}^p) \end{aligned}$$

Now, with $(**)$ yielding

$$\begin{aligned} (\mathbf{g}^j \cdot \mathbf{a}_k)(\mathbf{a}^k \cdot \mathbf{g}_p) &= (\mathbf{g}^j \otimes \mathbf{g}_p) \cdot (\mathbf{a}_k \otimes \mathbf{a}^k) = (\mathbf{g}^j \otimes \mathbf{g}_p) \cdot \mathbf{I} \\ &= (\mathbf{g}^j \otimes \mathbf{g}_p) \cdot (\mathbf{g}_n \otimes \mathbf{g}^n) = \delta_n^j\,\delta_p^n = \delta_p^j \end{aligned}$$

one finally obtains

$$\mathbf{T}\,\mathbf{T}^{-1} = \delta_p^j(\mathbf{g}_j \otimes \mathbf{g}^p) = (\mathbf{g}_p \otimes \mathbf{g}^p) = \mathbf{I} \qquad \text{q. e. d.}$$

In the special case, where the transformation of basis systems including a transition and a rotation reduces to a pure rotation, the metric coefficients of both basis systems coincide yielding

$$a_{ik} = \mathbf{a}_i \cdot \mathbf{a}_k = \mathbf{T}\,\mathbf{g}_i \cdot \mathbf{T}\,\mathbf{g}_k = \mathbf{g}_i \cdot \mathbf{g}_k = g_{ik}$$

In this case, the transformation reduces to a pure rotation $\mathbf{T} \equiv \mathbf{R}$ with $\mathbf{R}^{-1} = \mathbf{R}^T$, compare Section 2.4 (A).

Specific transformation for the determination of $\mathbf{a}_i$ as a function of the orthonormal basis $\mathbf{e}_i = \mathbf{e}^i$

$$\mathbf{a}_i = \mathbf{T}\,\mathbf{e}_i \quad \text{with} \quad \mathbf{T} = (\mathbf{e}^n \cdot \mathbf{a}_k)(\mathbf{e}_n \otimes \mathbf{e}^k)$$

such that

$$\mathbf{a}_i = [(\mathbf{e}^n \cdot \mathbf{a}_k)(\mathbf{e}_n \otimes \mathbf{e}^k)]\,\mathbf{e}_i = (\mathbf{a}_i \cdot \mathbf{e}^n)\,\mathbf{e}_n = (\mathbf{a}_n \cdot \mathbf{e}^i)\,\mathbf{e}_i$$

Equivalently, the inverse of the above transformation yields

$$\begin{aligned} &\mathbf{e}_i = \mathbf{T}^{-1}\mathbf{a}_i \quad \text{with} \quad \mathbf{T}^{-1} = (\mathbf{a}^n \cdot \mathbf{e}_k)(\mathbf{a}_n \otimes \mathbf{a}^k) \\ &\longrightarrow \quad \mathbf{e}_i = (\mathbf{e}_i \cdot \mathbf{a}^n)\,\mathbf{a}_n \end{aligned}$$

A.3 Vector or cross product of tangent vectors

Note: The cross product is also called “outer product”.

By use of the Ricci permutation tensor $\overset{3}{\mathbf{E}}$, compare Section 2.6, one obtains

$$\mathbf{a}_i \times \mathbf{a}_j = \overset{3}{\mathbf{E}}\,(\mathbf{a}_i \otimes \mathbf{a}_j)$$

Remark: Gregorio Ricci-Curbastro (1853-1925) was an Italian mathematician. Together with his scholar Tullio Levi-Civita (1873-1941), he is known as the father of tensor calculus.

Ricci **permutation tensor** $\overset{3}{\mathbf{E}}$

Note: Ricci's permutation tensor $\overset{3}{\mathbf{E}}$ is the fundamental tensor of 3rd order.

$$\boxed{\overset{3}{\mathbf{E}} = \begin{cases} \mathrm{E}_{ijk}\,\mathbf{a}^i \otimes \mathbf{a}^j \otimes \mathbf{a}^k \\ \mathrm{E}^{ijk}\,\mathbf{a}_i \otimes \mathbf{a}_j \otimes \mathbf{a}_k \end{cases}} \quad \text{with} \quad \begin{cases} \mathrm{E}_{ijk} = (\mathbf{a}_i \times \mathbf{a}_j) \cdot \mathbf{a}_k =: \sqrt{a}\;\mathrm{e}_{ijk} \\ \mathrm{E}^{ijk} = (\mathbf{a}^i \times \mathbf{a}^j) \cdot \mathbf{a}^k =: \sqrt{\bar{a}}\;\mathrm{e}^{ijk} \end{cases}$$

On the next page, it is shown that $\sqrt{a} = \sqrt{\det|a_{ik}|}$ and $\sqrt{\bar{a}} = \sqrt{\det|a^{ik}|} = (\sqrt{\det|a_{ik}|})^{-1}$.

Note: The products $(\mathbf{a}_i \times \mathbf{a}_j) \cdot \mathbf{a}_k$ or $(\mathbf{a}^i \times \mathbf{a}^j) \cdot \mathbf{a}^k$ also written as $[\mathbf{a}_i\,\mathbf{a}_j\,\mathbf{a}_k]$ or $[\mathbf{a}^i\,\mathbf{a}^j\,\mathbf{a}^k]$ are known as triple scalar or parallelepipedial products of the tangent and cotangent basis vectors. The permutation symbols e_{ijk} and e^{ijk} are known as Levi-Civita symbols.

Permutation symbols and orthonormal basis

$$\mathrm{e}_{ijk} = (\mathbf{e}_i \times \mathbf{e}_j) \cdot \mathbf{e}_k \quad \longrightarrow \quad \mathrm{e}_{ijk} = \mathrm{e}^{ijk} = \begin{cases} 1 & : \text{ even permutation} \\ -1 & : \text{ odd permutation} \\ 0 & : \text{ else} \end{cases}$$

$$\text{where} \quad \begin{cases} \mathrm{e}_{123} = (\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3 = \mathrm{e}_{231} = \mathrm{e}_{312} : & \text{even permutation} \\ \mathrm{e}_{321} = (\mathbf{e}_3 \times \mathbf{e}_2) \cdot \mathbf{e}_1 = \mathrm{e}_{213} = \mathrm{e}_{132} : & \text{odd permutation} \end{cases}$$

Therein, the specific form e_{123} of the general permutation symbol e_{ijk} represents the rectangular parallelepiped with side lengths 1 and volume 1.

Determinantion of a and $\bar{a}$ with the relations between E_{ijk} and E^{ijk}

$$\left. \begin{array}{lcl} \mathrm{E}_{ijk} =: \sqrt{a}\,\mathrm{e}_{ijk} & \longrightarrow & \mathrm{E}_{123} = \sqrt{a} \\ \mathrm{E}^{ijk} =: \sqrt{\bar{a}}\,\mathrm{e}^{ijk} & \longrightarrow & \mathrm{E}^{123} = \sqrt{\bar{a}} \end{array} \right\} \quad \longrightarrow \quad \frac{\mathrm{E}_{123}}{\mathrm{E}^{123}} = \frac{\sqrt{a}}{\sqrt{\bar{a}}}$$

By use of the metric coefficients, E^{ijk} can be displayed as function of E_{ijk} and vice versa:

$$\mathrm{E}^{stu} = a^{is}\, a^{jt}\, a^{ku}\, \mathrm{E}_{ijk} \quad \longrightarrow \quad \mathrm{E}^{123} = a^{i1}\, a^{j2}\, a^{k3}\, \mathrm{E}_{ijk}$$

$$\text{where} \quad \begin{cases} \mathrm{E}_{123} = \mathrm{E}_{231} = \mathrm{E}_{312} \\ \mathrm{E}_{321} = \mathrm{E}_{213} = \mathrm{E}_{132} \end{cases}$$

This yields

$$\mathrm{E}^{123} = [a^{11}\,(a^{22}a^{33} - a^{23}a^{32}) - a^{12}\,(a^{21}a^{33} - a^{23}a^{31}) + a^{13}\,(a^{21}a^{32} - a^{31}a^{22})]\, \mathrm{E}_{123}$$

$$\longrightarrow \quad \boxed{\mathrm{E}^{123} = \det |a^{ik}|\, \mathrm{E}_{123}}$$

The other way round shows that

$$\mathrm{E}_{stu} = a_{is}\, a_{jt}\, a_{ku}\, \mathrm{E}^{ijk} \quad \longrightarrow \quad \mathrm{E}_{123} = a_{i1}\, a_{j2}\, a_{k3}\, \mathrm{E}^{ijk}$$

$$\longrightarrow \quad \boxed{\mathrm{E}_{123} = \det |a_{ik}|\, \mathrm{E}^{123}}$$

Combining the above relations for E_{123} and E^{123}, one easily concludes to

$$\mathrm{E}_{123} = (\det |a_{ik}|)\, \mathrm{E}^{123} = (\det |a_{ik}|)\, (\det |a^{ik}|)\, \mathrm{E}_{123} \quad \longrightarrow \quad \det |a^{ik}| = (\det |a_{ik}|)^{-1}$$

$$\text{and} \quad \frac{\mathrm{E}_{123}}{\mathrm{E}^{123}} = \frac{\sqrt{a}}{\sqrt{\bar{a}}} = \det |a_{ik}| \quad \longrightarrow \quad a = \det |a_{ik}|\,, \;\; \bar{a} = \det |a^{ik}|$$

Determination of a and $\bar{a}$ with the cross product between basis vectors

$$\begin{aligned} \mathbf{a}_1 \times \mathbf{a}_2 &= \overset{3}{\mathbf{E}}\,(\mathbf{a}_1 \otimes \mathbf{a}_2) = \mathrm{E}_{ijk}\,(\mathbf{a}^i \otimes \mathbf{a}^j \otimes \mathbf{a}^k)(\mathbf{a}_1 \otimes \mathbf{a}_2) \\ &= \sqrt{a}\;\mathrm{e}_{ijk}\,\delta^j_1\,\delta^k_2\,\mathbf{a}^i = \sqrt{a}\;\mathrm{e}_{i12}\,\mathbf{a}^i = \sqrt{a}\;\mathrm{e}_{312}\,\mathbf{a}^3 \\ &= \sqrt{a}\;\mathbf{a}^3 \end{aligned}$$

Note: The vector or cross product between two objects, such as $\mathbf{a}_1$ and $\mathbf{a}_2$, is also called "outer product" as its result, a vector, is mapped into the respective dual space meaning that covariant objects yield a contravariant result and vice versa.

One obtains a vector in the direction of $\mathbf{a}^3$ with the value of $|\mathbf{a}_1 \times \mathbf{a}_2|$:

$$\begin{aligned} |\mathbf{a}_1 \times \mathbf{a}_2| &= |\mathbf{a}_1|\,|\mathbf{a}_2|\,\sin\varphi_{12} = \sqrt{a_{11}\,a_{22}\,\sin^2\varphi_{12}} \\ &= \sqrt{a_{11}\,a_{22}\,(1-\cos^2\varphi_{12})} \end{aligned}$$

where $\sin^2\varphi_{12} = 1 - \cos^2\varphi_{12}$ has been used. The value of the cosine function has been computed on p. 50, such that

$$\cos\varphi_{12} = \frac{a_{12}}{\sqrt{a_{11}\,a_{22}}} \qquad \longrightarrow \qquad a_{12} = \sqrt{a_{11}\,a_{22}}\,\cos\varphi_{12}$$

This leads to

$$|\mathbf{a}_1 \times \mathbf{a}_2| = \sqrt{a_{11}\,a_{22} - a_{11}\,a_{22}\,\cos^2\varphi_{12}} = \sqrt{a_{11}\,a_{22} - (a_{12})^2}$$

From a matrix $A = |a_{ik}|$ with determinant

$$\det A = \det|a_{ik}| = \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

it is seen that $a_{11}\,a_{22} - (a_{12})^2$ is the upper left subdeterminant of $\det a_{ik}$ and the entry $\operatorname{cof} a_{33} = (\operatorname{cof} a_{33})^T$ of the cofactor $\operatorname{cof} A$ of A at the position $(\,\cdot\,)_{33}$. With this information, the cross product of $\mathbf{a}_1$ and $\mathbf{a}_2$ reads

$$\mathbf{a}_1 \times \mathbf{a}_2 = |\mathbf{a}_1 \times \mathbf{a}_2|\,\frac{\mathbf{a}^3}{|\mathbf{a}^3|} = \frac{\sqrt{a_{11}\,a_{22} - (a_{12})^2}}{\sqrt{a^{33}}}\,\mathbf{a}^3 = \frac{\sqrt{\operatorname{cof} a_{33}}}{\sqrt{a^{33}}}\,\mathbf{a}^3$$

In the next step, a^{33} has to be found. Assume a^{33} as an entry of $A^{-1} = |a^{ik}|$ inverse to $A = |a_{ik}|$, such that one obtains A^{-1} via

$$A^{-1} = \frac{(\operatorname{cof} A)^T}{\det A} \qquad \longrightarrow \qquad a^{33} = \frac{\operatorname{cof} a_{33}}{\det A}$$

Now, it follows from the above that

$$\begin{aligned} &\mathbf{a}_1 \times \mathbf{a}_2 = \mathrm{E}_{123}\,\mathbf{a}^3 = \sqrt{\operatorname{cof} a_{33}\frac{\det|a_{ik}|}{\operatorname{cof} a_{33}}}\,\mathbf{a}^3 = \sqrt{\det|a_{ik}|}\,\mathbf{a}^3 = \sqrt{\det A}\,\mathbf{a}^3 \\ &\longrightarrow \quad \mathrm{E}_{123} = \sqrt{\det A} = \sqrt{\det|a_{ik}|} = \sqrt{a} \end{aligned}$$

Analogously, one finds

$$\begin{aligned} &\mathbf{a}^1 \times \mathbf{a}^2 = \mathrm{E}^{123}\,\mathbf{a}_3 = \sqrt{\det|a^{ik}|}\,\mathbf{a}_3 = \sqrt{\det A^{-1}}\,\mathbf{a}_3 = \sqrt{(\det A)^{-1}}\,\mathbf{a}_3 \\ &\longrightarrow \quad \mathrm{E}^{123} = \sqrt{(\det A)^{-1}} = \sqrt{(\det|a_{ik}|)^{-1}} = \frac{1}{\sqrt{a}} \end{aligned}$$

with $\bar{a} = a^{-1}$, where $a = \det|a_{ik}|$ and $a^{-1} = (\det|a_{ik}|)^{-1} = \det|a^{ik}| = \bar{a}$.

Note: As RICCI's permutation tensor $\overset{3}{\mathbf{E}}$ is a fundamental tensor only formed by basis vectors, the products of $\overset{3}{\mathbf{E}}$ are identical whatever basis is chosen, compare Section 2.6:

$$\overset{3}{\mathbf{E}} \cdot \overset{3}{\mathbf{E}} = 6\,, \quad (\overset{3}{\mathbf{E}}\overset{3}{\mathbf{E}})^{\underline{2}} = 2\,\mathbf{I}\,, \quad (\overset{3}{\mathbf{E}}\overset{3}{\mathbf{E}})^{\underline{4}} = (\mathbf{I} \otimes \mathbf{I})^{\overset{23}{T}} - (\mathbf{I} \otimes \mathbf{I})^{\overset{24}{T}}$$

To prove this feature by use of the natural basis system, the first identity reads

$$\begin{aligned}
\overset{3}{\mathbf{E}} \cdot \overset{3}{\mathbf{E}} &= \mathrm{E}_{ijk}\,(\mathbf{a}^i \otimes \mathbf{a}^j \otimes \mathbf{a}^k) \cdot \mathrm{E}_{sto}\,(\mathbf{a}^s \otimes \mathbf{a}^t \otimes \mathbf{a}^o) \\
&= \mathrm{E}_{ijk}\,\mathrm{E}_{sto}\,a^{is}\,a^{jt}\,a^{ko} = \mathrm{E}_{ijk}\,\mathrm{E}^{ijk} \\
&= \sqrt{a}\,\mathrm{e}_{ijk}\,(\sqrt{a})^{-1}\mathrm{e}^{ijk} = \mathrm{e}_{ijk}\,\mathrm{e}^{ijk} = 6
\end{aligned}$$

From the above result, an additional relation for the determinant $a = \det A$ of metric coefficients can be seen:

$$\begin{aligned}
6 &= \mathrm{E}_{ijk}\,\mathrm{E}_{sto}\,a^{is}\,a^{jt}\,a^{ko} = \sqrt{a}\,\mathrm{e}_{ijk}\,\sqrt{a}\,\mathrm{e}_{sto}\,a^{is}\,a^{jt}\,a^{ko} \\
&= \det A\,\mathrm{e}_{ijk}\;\mathrm{e}_{sto}\,a^{is}\,a^{jt}\,a^{ko} \quad\longrightarrow\quad (\det A)^{-1} = \tfrac{1}{6}\,\mathrm{e}_{ijk}\;\mathrm{e}_{sto}\,a^{is}\,a^{jt}\,a^{ko}
\end{aligned}$$

Equivalently, one also obtains $\boxed{\det A = \tfrac{1}{6}\,\mathrm{e}^{ijk}\;\mathrm{e}^{sto}\,a_{is}\,a_{jt}\,a_{ko}}$

The second identity reads

$$\begin{aligned}
(\overset{3}{\mathbf{E}}\,\overset{3}{\mathbf{E}})^{\underline{2}} &= \mathrm{E}_{ijk}\,\mathrm{E}_{sto}\,[\,(\mathbf{a}^i \otimes \mathbf{a}^j \otimes \mathbf{a}^k)\,(\mathbf{a}^s \otimes \mathbf{a}^t \otimes \mathbf{a}^o)\,]^{\underline{2}} \\
&= \mathrm{E}_{ijk}\,\mathrm{E}_{sto}\,a^{js}\,a^{kt}\,(\mathbf{a}^i \otimes \mathbf{a}^o) = \mathrm{E}_{ijk}\,\mathrm{E}_{sto}\,a^{js}\,a^{kt}\,(\mathbf{a}^i \otimes\; a^{op}\mathbf{a}_p) \\
&= \mathrm{E}_{ijk}\,\mathrm{E}^{jkp}\,(\mathbf{a}^i \otimes \mathbf{a}_p) = \mathrm{e}_{ijk}\,\mathrm{e}^{jkp}\,(\mathbf{a}^i \otimes \mathbf{a}_p) = 2\,\mathbf{I}
\end{aligned}$$

The final identity is obtained via

$$\begin{aligned}
(\overset{3}{\mathbf{E}}\,\overset{3}{\mathbf{E}})^{\underline{4}} &= \mathrm{E}_{ijk}\,\mathrm{E}_{sto}\,[\,(\mathbf{a}^i \otimes \mathbf{a}^j \otimes \mathbf{a}^k)\,(\mathbf{a}^s \otimes \mathbf{a}^t \otimes \mathbf{a}^o)\,]^{\underline{4}} \\
&= \mathrm{E}_{ijk}\,\mathrm{E}_{sto}\,a^{ks}\,(\mathbf{a}^i \otimes \mathbf{a}^j \otimes \mathbf{a}^t \otimes \mathbf{a}^o) \\
&= \mathrm{E}_{ijk}\,\mathrm{E}_{sto}\,a^{ks}\,a^{tp}\,a^{or}\,(\mathbf{a}^i \otimes \mathbf{a}^j \otimes \mathbf{a}_p \otimes \mathbf{a}_r) \\
&= \mathrm{E}_{ijk}\,\mathrm{E}^{kpr}\,(\mathbf{a}^i \otimes \mathbf{a}^j \otimes \mathbf{a}_p \otimes \mathbf{a}_r) = \mathrm{e}_{ijk}\,\mathrm{e}^{kpr}\,(\mathbf{a}^i \otimes \mathbf{a}^j \otimes \mathbf{a}_p \otimes \mathbf{a}_r) = \;... \\
&= (\mathbf{I} \otimes \mathbf{I})^{\overset{23}{T}} - (\mathbf{I} \otimes \mathbf{I})^{\overset{24}{T}}
\end{aligned}$$

Determinant of an arbitrary tensor T

From the basic rules of tensor calculus, the determinant of $\mathbf{T} = t^{ik}(\mathbf{a}_i \otimes \mathbf{a}_k)$ is defined by the outer tensor product of tensors, compare Section **2.8**, via

$$\det \mathbf{T} = \tfrac{1}{6}(\mathbf{T} \circledast \mathbf{T}) \cdot \mathbf{T}$$

This rule will be computed step by step:

$$\begin{aligned} \mathbf{T} \mathbin{⨳} \mathbf{T} &= t^{ik}\, t^{no}\, (\mathbf{a}_i \otimes \mathbf{a}_k) \mathbin{⨳} (\mathbf{a}_n \otimes \mathbf{a}_o) \\ &= t^{ik}\, t^{no}\, (\mathbf{a}_i \times \mathbf{a}_n) \otimes (\mathbf{a}_k \times \mathbf{a}_o) \end{aligned}$$

With $(\mathbf{a}_i \times \mathbf{a}_n) = \mathrm{E}_{ins}\, \mathbf{a}^s = \sqrt{a}\, \mathrm{e}_{ins}\, \mathbf{a}^s$ and $(\mathbf{a}_k \times \mathbf{a}_o) = \sqrt{a}\, \mathrm{e}_{kop}\, \mathbf{a}^p$, the above equation reads

$$\begin{aligned} \mathbf{T} \mathbin{⨳} \mathbf{T} &= t^{ik}\, t^{no} (\sqrt{a}\, \mathrm{e}_{ins}\, \mathbf{a}^s) \otimes (\sqrt{a}\, \mathrm{e}_{kop}\, \mathbf{a}^p) \\ &= a\ t^{ik}\, t^{no} \mathrm{e}_{ins} \mathrm{e}_{kop}\, (\mathbf{a}^s \otimes \mathbf{a}^p) \end{aligned}$$

By use of the matrix $A = |a_{ik}|$ with $\det A = a$, the next step yields the determinant of $\mathbf{T}$ reading

$$\begin{aligned} \det \mathbf{T} &= \det A\, \tfrac{1}{6} \left[t^{ik}\, t^{no} \mathrm{e}_{ins} \mathrm{e}_{kop}\, (\mathbf{a}^s \otimes \mathbf{a}^p) \cdot t^{rm}\, (\mathbf{a}_r \otimes \mathbf{a}_m) \right] \\ &= \det A\, \tfrac{1}{6} \left[t^{ik}\, t^{no}\, t^{rm} \mathrm{e}_{ins} \mathrm{e}_{kop}\, \delta^s_r\, \delta^p_m \right] \\ &= \det A\, \tfrac{1}{6} \left[t^{ik}\, t^{no}\, t^{sp} \mathrm{e}_{ins} \mathrm{e}_{kop} \right] =: (\det T)\, (\det A) \end{aligned}$$

with $\det T = \det |t^{ik}|$ and $\det A = \det |a_{ik}|$. By a lengthy computation resulting in

$$\begin{aligned} \tfrac{1}{6} \left[t^{ik}\, t^{no}\, t^{sp} \mathrm{e}_{ins} \mathrm{e}_{kop} \right] &= \tfrac{1}{6} \{ t^{11}\, 6 \left[(t^{22} t^{33} - t^{23} t^{32}) + t^{12}\, 6 \left[(t^{23} t^{31} - t^{21} t^{33}) \right] \right. \\ &\quad \left. + t^{13}\, 6 \left[(t^{21} t^{32} - t^{22} t^{31}) \right] \right\} \end{aligned}$$

one obtains the following rule

$$\boxed{\begin{aligned} &\det \mathbf{T} = \det \left[t^{ik} (\mathbf{a}_i \otimes \mathbf{a}_k) \right] = (\det T)\, (\det A) \\ &\text{with} \quad \begin{cases} \det T = \det |t^{ik}| = \tfrac{1}{6} \left[t^{ik}\, t^{no}\, t^{sp}\, \mathrm{e}_{ins} \mathrm{e}_{kop} \right] \\ \det A = \det |a_{ik}| = \tfrac{1}{6} \left[a_{ik}\, a_{no}\, a_{sp}\, \mathrm{e}^{ins} \mathrm{e}^{kop} \right] \end{cases} \end{aligned}}$$

In addition to the above, one obtains for contra- and mixedvariant $\mathbf{T}$ that

$$\begin{aligned} \det \mathbf{T} &= \det \left[t_{ik} (\mathbf{a}^i \otimes \mathbf{a}^k) \right] \\ &= (\det \bar{T})\, (\det \bar{A}) \end{aligned} \qquad \text{with} \quad \begin{cases} \det \bar{T} = \det |t_{ik}| \\ \det \bar{A} = \det |a^{ik}| = \det |\mathbf{a}^i \cdot \mathbf{a}^k| \end{cases}$$

$$\begin{aligned} \det \mathbf{T} &= \det \left[t_i{}^k (\mathbf{a}^i \otimes \mathbf{a}_k) \right] \\ &= (\det \tilde{T})\, (\det \tilde{A}) \end{aligned} \qquad \text{with} \quad \begin{cases} \det \tilde{T} = \det |t_i{}^k| \\ \det \tilde{A} = \det |\delta^i_k| = \det |\mathbf{a}^i \cdot \mathbf{a}_k| = 1 \end{cases}$$

From the above rule, it obvious that the determinant of a tensor $\mathbf{T} = t^{ik} (\mathbf{e}_i \otimes \mathbf{e}_k)$ given in an orthonormal basis $(\mathbf{e}_i \otimes \mathbf{e}_k)$ is equivalent to the determinant of its coefficient matrix yielding $\det \mathbf{T} = \det |t^{ik}|$ as $\det A = \det |\delta_{ik}| = 1$.

Examples: Vector product of arbitrary vectors $\mathbf{u}$ and $\mathbf{v}$

$$\mathbf{u} \times \mathbf{v} = |\mathbf{u}|\, |\mathbf{v}|\, \sin \sphericalangle(\mathbf{u}\, ;\, \mathbf{v})\, \mathbf{n}$$

with $\sin \sphericalangle(\mathbf{u}\,;\mathbf{v}) = \cos(1/2\,\pi - \sphericalangle(\mathbf{u}\,;\mathbf{v}))$ and $\mathbf{n}$: unit vector $\perp\{\mathbf{u},\ \mathbf{v}\}$, following the corkscrew or right-hand rule on page 7.

Norm of the vector product

$$|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\,\sin\sphericalangle(\mathbf{u}\,;\mathbf{v})$$

By use of $\overset{3}{\mathbf{E}}$, the vector product between $\mathbf{u}$ and $\mathbf{v}$ can be obtained as

$$\begin{aligned}
\mathbf{u}\times\mathbf{v} &= u^i\,\mathbf{a}_i \times v^k\,\mathbf{a}_k \;=\; u^i v^k\,\overset{3}{\mathbf{E}}\,(\mathbf{a}_i\otimes\mathbf{a}_k) \;=\; u^i v^k\,\mathrm{E}_{nop}\,(\mathbf{a}^n\otimes\mathbf{a}^o\otimes\mathbf{a}^p)(\mathbf{a}_i\otimes\mathbf{a}_k)\\
&= u^i v^k\,\sqrt{a}\;\mathrm{e}_{nop}\,\delta^o_i\,\delta^p_k\mathbf{a}^n \;=\; u^i v^p\,\sqrt{a}\;\mathrm{e}_{nip}\,\mathbf{a}^n\\
&= \sqrt{a}\;[(u^2\,v^3 - u^3\,v^2)\,\mathbf{a}^1 + (u^3\,v^1 - u^1\,v^3)\,\mathbf{a}^2 + (u^1\,v^2 - u^2\,v^1)\,\mathbf{a}^3\,]
\end{aligned}$$

Scalar triple product (parallelepipedial product)

$$\begin{aligned}
(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} &= (\mathbf{v}\times\mathbf{w})\cdot\mathbf{u} \;=\; (\mathbf{w}\times\mathbf{u})\cdot\mathbf{v}\\
&= (u^i\,\mathbf{a}_i\times v^j\,\mathbf{a_j})\cdot w^k\,\mathbf{a}_k \;=\; u^i\,v^j\,w^k\,(\mathbf{a}_i\times\mathbf{a_j})\cdot\mathbf{a}_k\\
&= E_{ijk}\,u^i\,v^j\,w^k \;=\; \sqrt{a}\,e_{ijk}\,u^i\,v^j\,w^k\\
&= u^1\,(v^2\,w^3 - v^3\,w^2)\underbrace{+\;u^2\,(v^3\,w^1 - v^1\,w^3)}_{-u^2\,(v^1\,w^3 - v^3\,w^1)} + u^3\,(v^1\,w^2 - v^2\,w^1)
\end{aligned}$$

A.4 Spatial derivative of natural basis systems

Given curved parameter lines θ^i, the natural basis vectors $\mathbf{a}_i$ change their values and directions along θ^i. This makes it necessary to include spatial derivatives of tangent and cotangent basis vectors.

(a) Derivatives of tangent vectors $\mathbf{a}_i$ and Christoffel symbols of the 2nd kind

Procedure: One forms spatial derivatives $\mathbf{a}_{i,k}$ and applies them with the help of CHRISTOFFEL symbols to the tangent basis $\mathbf{a}_k$

$$\boxed{\mathbf{a}_{i,k} := \frac{\partial\mathbf{a}_i}{\partial\theta^j} =: \Gamma_{ij}{}^k\mathbf{a}_k}$$

with $\Gamma_{ij}{}^k$: CHRISTOFFEL symbols of the 2nd kind

Elwin Bruno CHRISTOFFEL (1829-1900) was a German mathematician and physicist. He introduced fundamental concepts of differential geometry, thus opening the way for the development of tensor calculus.

CHRISTOFFEL **symbols of the 2nd kind**

$$\mathbf{a}_{i,j}\cdot\mathbf{a}^s = \Gamma_{ij}{}^k\mathbf{a}_k\,\cdot\mathbf{a}^s =: \Gamma_{ij}{}^k\,\delta^s_k = \Gamma_{ij}{}^s \quad\longrightarrow\quad \boxed{\Gamma_{ij}{}^s = \mathbf{a}_{i,j}\cdot\mathbf{a}^s}$$

Transformation of CHRISTOFFEL symbols

$$\delta^i_s = \left\{ \begin{array}{l} 1 \\ 0 \end{array} \right. \longrightarrow \quad (\delta^i_s)_{,j} = 0 \quad \text{with} \quad \delta^i_s = \mathbf{a}^i \cdot \mathbf{a}_s$$

$$\longrightarrow \quad 0 = (\mathbf{a}^i \cdot \mathbf{a}_s)_{,j} = \mathbf{a}^i{}_{,j} \cdot \mathbf{a}_s + \mathbf{a}^i \cdot \mathbf{a}_{s,j}$$

Thus, it follows that

$$\underbrace{\mathbf{a}_{s,j} \cdot \mathbf{a}^i}_{\Gamma_{sj}{}^i} = -\underbrace{\mathbf{a}^i{}_{,j} \cdot \mathbf{a}_s}_{\Gamma^i{}_{js}} \quad \longrightarrow \quad \boxed{\Gamma_{sj}{}^i = -\Gamma^i{}_{js}}$$

(b) Derivatives of cotangent vectors $\mathbf{a}^i$ and CHRISTOFFEL symbols of the 1st kind

Procedure: One forms spatial derivatives $\mathbf{a}^i{}_{,j}$ and applies them with the help of CHRISTOFFEL symbols to the cotangent basis $\mathbf{a}^k$

$$\boxed{\mathbf{a}^i{}_{,j} := \frac{\partial \mathbf{a}^i}{\partial \theta^j} =: \Gamma^i{}_{jk}\, \mathbf{a}^k = -\Gamma_{kj}{}^i \mathbf{a}^k}$$

with $\Gamma^i{}_{jk} = \mathbf{a}^i{}_{,j} \cdot \mathbf{a}_k = -\Gamma_{kj}{}^i = \mathbf{a}_{k,j} \cdot \mathbf{a}^i$.

CHRISTOFFEL **symbols of the 1st kind**

$$\mathbf{a}_{i,j} := \Gamma_{ij}{}^k\, \mathbf{a}_k = \Gamma_{ij}{}^k\, a_{ks}\, \mathbf{a}^s = \Gamma_{ijs}\, \mathbf{a}^s$$

with Γ_{ijs}: CHRISTOFFEL symbols of the 1st kind

Determination of Γ_{ijs}

$$\mathbf{a}_{i,j} \cdot \mathbf{a}_k = \Gamma_{ijs}\, \mathbf{a}^s \cdot \mathbf{a}_k = \Gamma_{ijs}\, \delta^s_k = \Gamma_{ijk} \quad \longrightarrow \quad \boxed{\Gamma_{ijk} = \mathbf{a}_{i,j} \cdot \mathbf{a}_k}$$

Note: The super- and subscripts of CHRISTOFFEL symbols can only be pulled up and down by metric coefficients as far as they are **not** in connection with spatial derivatives.

A.5 Gradient and divergence operators

Gradient of a scalar-valued function $\phi(\mathbf{x})$

$$\operatorname{grad} \phi(\mathbf{x}) = \frac{\mathrm{d}\phi}{\mathrm{d}\mathbf{x}} = \frac{\partial \phi}{\partial \theta^i} \frac{\partial \theta^i}{\partial \mathbf{x}} = \phi_{,i}\, \mathbf{a}^i$$

Gradient of a vector-valued function $\mathbf{v}(\mathbf{x})$

$$\operatorname{grad} \mathbf{v}(\mathbf{x}) = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\mathbf{x}} = \frac{\partial \mathbf{v}}{\partial \theta^i} \otimes \frac{\partial \theta^i}{\partial \mathbf{x}} = \mathbf{v}_{,i} \otimes \mathbf{a}^i$$

Given $\mathbf{v}$ in a covariant basis, one obtains with the CHRISTOFFEL symbols of the 2nd kind

$$\mathbf{v}_{,i} = (v^n \, \mathbf{a}_n)_{,i} = v^n{}_{,i} \, \mathbf{a}_n + v^n \, \mathbf{a}_{n,i} = v^n{}_{,i} \, \mathbf{a}_n + v^n \, \Gamma_{ni}{}^s \, \mathbf{a}_s = v^n{}_{,i} \, \mathbf{a}_n + v^s \, \Gamma_{si}{}^n \, \mathbf{a}_n$$

Note that the last term has been obtained by renaming the indices s to n and n to s. Thus,

$$\boxed{\mathbf{v}_{,i} = (v^n{}_{,i} + v^s \, \Gamma_{si}{}^n) \, \mathbf{a}_n =: v^n\big|_i \, \mathbf{a}_n}$$

with $\quad v^n\big|_i$: covariant derivative

Gradients of vector-valued functions $\mathbf{v}(\mathbf{x})$

$$\boxed{\operatorname{grad}\mathbf{v} \begin{cases} = & v^n\big|_i \, \mathbf{a}_n \otimes \mathbf{a}^i \\ = & v_n\big|_i \, \mathbf{a}^n \otimes \mathbf{a}^i \end{cases}}$$

Divergence of a vector-valued function $\mathbf{v}(\mathbf{x})$

$$\begin{aligned} \operatorname{div}\mathbf{v} &= \operatorname{grad}\mathbf{v} \cdot \mathbf{I} \\ &= v^n\big|_i \, (\mathbf{a}_n \otimes \mathbf{a}^i) \cdot (\mathbf{a}^j \otimes \mathbf{a}_j) \\ &= v^n\big|_i \, \delta^j_n \, \delta^i_j = v^i\big|_i \end{aligned}$$

Gradient of a tensor-valued function $\mathbf{T}(\mathbf{x})$

$$\operatorname{grad}\mathbf{T} = \frac{\partial \mathbf{T}}{\partial \theta^i} \otimes \frac{\partial \theta^i}{\partial \mathbf{x}} = \mathbf{T}_{,n} \otimes \mathbf{a}^n$$

Given a covariant basis of $\mathbf{T}$, one obtains with the CHRISTOFFEL symbols of the 2nd kind

$$\begin{aligned} \mathbf{T}_{,n} &= (t^{ij} \mathbf{a}_i \otimes \mathbf{a}_j)_{,n} \\ &= t^{ij}{}_{,n} \, \mathbf{a}_i \otimes \mathbf{a}_j + t^{ij} (\underbrace{\mathbf{a}_{i,n}}_{\Gamma_{in}{}^s \, \mathbf{a}_s} \otimes \, \mathbf{a}_j + \mathbf{a}_i \, \otimes \, \underbrace{\mathbf{a}_{j,n}}_{\Gamma_{jn}{}^s \, \mathbf{a}_s}) \\ &= t^{ij}{}_{,n} \, \mathbf{a}_i \otimes \mathbf{a}_j + t^{sj} \Gamma_{sn}{}^i \, \mathbf{a}_i \otimes \, \mathbf{a}_j + t^{is} \Gamma_{sn}{}^j \, \mathbf{a}_i \, \otimes \mathbf{a}_j \end{aligned}$$

Note that in the last line of $\mathbf{T}_{,n}$ use has been made of renaming i to s and s to i in the 2nd term and renaming j to s and s to j in the 3rd term. Thus, one obtains

$$\boxed{\mathbf{T}_{,n} = \underbrace{(t^{ij}{}_{,n} + t^{sj} \Gamma_{sn}{}^i + t^{is} \Gamma_{sn}{}^j)}_{t^{ij}\big|_n} (\mathbf{a}_i \otimes \mathbf{a}_j) = t^{ij}\big|_n \mathbf{a}_i \otimes \mathbf{a}_j}$$

In case of a contravariant basis of $\mathbf{T}$, the same procedure as above yields

$$\mathbf{T}_{,n} = \underbrace{\left(t_{ij,n} - t_{sj}\Gamma_{in}{}^{s} - t_{is}\Gamma_{jn}{}^{s}\right)}_{t_{ij}\big|_n}\left(\mathbf{a}^i \otimes \mathbf{a}^j\right) \; = \; t_{ij}\big|_n \mathbf{a}^i \otimes \mathbf{a}^j$$

With the above information, the gradient of $\mathbf{T}$ results in

$$\operatorname{grad}\mathbf{T} = \begin{cases} t^{ij}\big|_n \mathbf{a}_i \otimes \mathbf{a}_j \otimes \mathbf{a}^n \\ t_{ij}\big|_n \mathbf{a}^i \otimes \mathbf{a}^j \otimes \mathbf{a}^n \end{cases}$$

Divergence of a tensor-valued function $\mathbf{T}(\mathbf{x})$

$$\begin{aligned} \operatorname{div}\mathbf{T} &= (\operatorname{grad}\mathbf{T})\,\mathbf{I} \\ &= t^{ij}\big|_n (\mathbf{a}_i \otimes \mathbf{a}_j \otimes \mathbf{a}^n)\,(\mathbf{a}^s \otimes \mathbf{a}_s) \; = \; t^{ij}\big|_n \delta^s_j \, \delta^n_s \mathbf{a}_i \end{aligned}$$

$$\longrightarrow \quad \boxed{\operatorname{div}\mathbf{T} = t^{ij}\big|_j \mathbf{a}_i}$$

B Geometric measures of solid mechanics

B.1 Deformation gradient and deformation tensors

In solid mechanics, the motion of a body resulting from external forces and temperature changes is described in a LAGRANGEan setting, where the deformation gradient $\mathbf{F}$ relates the motion function $\boldsymbol{\chi}(\mathbf{x}_0, t)$ of material points of the current position $\mathbf{x}$ at time $t > t_0$ to their reference position $\mathbf{x}_0$ at time t_0.

Joseph-Louis LAGRANGE (1736–1813) was an Italian-French mathematician and astronomer, who later became a naturalized French. He made significant contributions to the fields of analysis, number theory, and both classical and celestial mechanics.

Based on arbitrary parameter lines θ^i, $\mathbf{F}$ is governed by

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\mathbf{x}_0} = \frac{\partial \mathbf{x}}{\partial \theta^i} \otimes \frac{\partial \theta^i}{\partial \mathbf{x}_0} =: \mathbf{a}_i \otimes \mathbf{h}^i$$

Therein, $\mathbf{a}_i$ is the tangential natural basis vector in terms of the current configuration at t, while $\mathbf{h}^i$ is the cotangential dual basis vector at the reference configuration at t_0. Thus, $\mathbf{F}$ is a two-field tensor with one basis system in the current and the other basis system in the reference configuration.

Given $\mathbf{F}$, one computes $\mathbf{F}^{-1}$ as

$$\mathbf{F}^{-1} = \frac{\mathrm{d}\mathbf{x}_0}{\mathrm{d}\mathbf{x}} = \frac{\partial \mathbf{x}_0}{\partial \theta^i} \otimes \frac{\partial \theta^i}{\partial \mathbf{x}} =: \mathbf{h}_i \otimes \mathbf{a}^i$$

Note that $\mathbf{F}$ and $\mathbf{F}^{-1}$ can also be understood as transport mechanisms by the property

$$\begin{array}{rclcrcll} \mathbf{F}\,\mathbf{h}_i &=& \mathbf{a}_i & \text{and} & \mathbf{F}^{-1}\mathbf{a}_i &=& \mathbf{h}_i & : \text{ covariant push forward and pull back} \\ \mathbf{F}^{T-1}\mathbf{h}^i &=& \mathbf{a}^i & \text{and} & \mathbf{F}^{T}\mathbf{a}^i &=& \mathbf{h}^i & : \text{ contravariant push forward and pull back} \end{array}$$

With $\mathbf{F}$, one computes the right and left CAUCHY-GREEN deformation tensors as

$$\begin{array}{ll} \mathbf{C} = \mathbf{F}^T\mathbf{F} = (\mathbf{h}^i \otimes \mathbf{a}_i)(\mathbf{a}_j \otimes \mathbf{h}^j) = a_{ij}(\mathbf{h}^i \otimes \mathbf{h}^j) & : \text{ right CAUCHY-GREEN} \\ \mathbf{B} = \mathbf{F}\mathbf{F}^T = (\mathbf{a}_i \otimes \mathbf{h}^i)(\mathbf{h}^j \otimes \mathbf{a}_j) = h^{ij}(\mathbf{a}_i \otimes \mathbf{a}_j) & : \text{ left CAUCHY-GREEN} \end{array}$$

Remark: Augustin-Louis CAUCHY (1789-1857) was a French mathematician and professor at the École polytechnique at Paris, while George GREEN (1793-1841) was an English miller and self-taught mathematician. His work was only found important by William Thomson (Lord Kelvin) four years after his death.

As $\mathbf{C}$ is contravariant while $\mathbf{B}$ is covariant, $\mathbf{B}^{-1}$ and $\mathbf{C}^{-1}$ enter the stage as further deformation tensors with inverse variances compared to $\mathbf{C}$ and $\mathbf{B}$:

$$\begin{array}{rcl} \mathbf{C}^{-1} &=& \mathbf{F}^{-1}\mathbf{F}^{T-1} = (\mathbf{h}_i \otimes \mathbf{a}^i)\,(\mathbf{a}^j \otimes \mathbf{h}_j) = a^{ij}\,\mathbf{h}_i \otimes \mathbf{h}_j \\ \mathbf{B}^{-1} &=& \mathbf{F}^{T-1}\mathbf{F}^{-1} = (\mathbf{a}^i \otimes \mathbf{h}_i)\,(\mathbf{h}_j \otimes \mathbf{a}^j) = h_{ij}\,\mathbf{a}^i \otimes \mathbf{a}^j \end{array}$$

B.2 Co- and contravariant strain tensors

Based on the terms above, the basic strain tensors (contravariant strains) yield

$$\begin{array}{rclcll} \mathbf{E} &=& \frac{1}{2}(\mathbf{C} - \mathbf{I}) &=& \frac{1}{2}(a_{ij} - h_{ij})(\mathbf{h}^i \otimes \mathbf{h}^j) & : \text{ GREEN-LAGRANGE} \\ \mathbf{A} &=& \frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1}) &=& \frac{1}{2}(a_{ij} - h_{ij})(\mathbf{a}^i \otimes \mathbf{a}^j) & : \text{ ALMANSI} \end{array}$$

Emilio ALMANSI (1869-1948) was an Italian mathematician.

Note that $\mathbf{E}$ is based on the reference configuration with contravariant basis vectors $\mathbf{h}^i$ that are constant over time, while $\mathbf{A}$ depends on contravariant basis vectors $\mathbf{a}^i$ of the current configuration that change over time. Thus, the strain is basically stored in the difference between a_{ij} and h_{ij}, while $\mathbf{A}$ has a further contribution through the basis $\mathbf{a}_i$.

The GREEN-LAGRANGE and ALMANSI strains are connected to each other by contravariant push-forward and pull-back transformations reading

$$\begin{array}{rclcl} \mathbf{A} &=& \mathbf{F}^{T-1}\mathbf{E}\,\mathbf{F}^{-1} = \frac{1}{2}(a_{ij} - h_{ij})\,\underbrace{\mathbf{F}^{T-1}(\mathbf{h}^i \otimes \mathbf{h}^j)\mathbf{F}^{-1}}_{\mathbf{F}^{T-1}\mathbf{h}^i \otimes \mathbf{F}^{T-1}\mathbf{h}^j} &=& \frac{1}{2}(a_{ij} - h_{ij})\,\mathbf{a}^i \otimes \mathbf{a}^j \\ \mathbf{E} &=& \mathbf{F}^T\mathbf{A}\,\mathbf{F} = \frac{1}{2}(a_{ij} - h_{ij})\,\underbrace{\mathbf{F}^T(\mathbf{a}^i \otimes \mathbf{a}^j)\mathbf{F}}_{\mathbf{F}^T\mathbf{a}^i \otimes \mathbf{F}^T\mathbf{a}^j} &=& \frac{1}{2}(a_{ij} - h_{ij})\,\mathbf{h}^i \otimes \mathbf{h}^j \end{array}$$

In addition to the above, there is a further set of strain tensors, the covariant KARNI-REINER strains, yielding

$$
\boxed{\begin{array}{llll}
\mathbf{K}_R = \frac{1}{2}(\mathbf{I} - \mathbf{C}^{-1}) & = \frac{1}{2}(h^{ij} - a^{ij})(\mathbf{h}_i \otimes \mathbf{h}_j) & : & \text{reference-configuration-based} \\
\mathbf{K}_C = \frac{1}{2}(\mathbf{B} - \mathbf{I}) & = \frac{1}{2}(h^{ij} - a^{ij})(\mathbf{a}_i \otimes \mathbf{a}_j) & : & \text{current-configuration-based}
\end{array}}
$$

Zvi KARNI and Markus REINER have been working at the Israel Institute of Technology, Haifa, Israel. They published serveral papers in the 50th and 60th of the last century.

Here, the KARNI-REINER strains are connected to each other by covariant push-forward and pull-back transformations yielding

$$
\begin{aligned}
\mathbf{K}_C &= \mathbf{F}\,\mathbf{K}_R\mathbf{F}^T = \tfrac{1}{2}(h^{ij} - a^{ij})\,\underbrace{\mathbf{F}(\mathbf{h}_i \otimes \mathbf{h}_j)\mathbf{F}^T}_{\mathbf{F}\,\mathbf{h}_i \otimes \mathbf{F}\,\mathbf{h}_j} = \tfrac{1}{2}(h^{ij} - a^{ij})\,\mathbf{a}_i \otimes \mathbf{a}_j \\
\mathbf{K}_R &= \mathbf{F}^{-1}\,\mathbf{K}_C\mathbf{F}^{T-1} = \tfrac{1}{2}(h^{ij} - a^{ij})\,\underbrace{\mathbf{F}^{-1}(\mathbf{a}_i \otimes \mathbf{a}_j)\mathbf{F}^{T-1}}_{\mathbf{F}^{-1}\,\mathbf{a}_i \otimes \mathbf{F}^{-1}\,\mathbf{a}_j} = \tfrac{1}{2}(h^{ij} - a^{ij})\,\mathbf{h}_i \otimes \mathbf{h}_j
\end{aligned}
$$

B.3 Deformation and strain velocities

Material velocity gradient $\dot{\mathbf{F}}$: Based on the deformation gradient $\mathbf{F}$, the material velocity gradient reads

$$
\dot{\mathbf{F}} = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}\Big) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{a}_i \otimes \mathbf{h}^i) = \dot{\mathbf{a}}_i \otimes \mathbf{h}^i
$$

Spatial velocity gradient $\mathbf{L}$: Once the material deformation velocity $\dot{\mathbf{F}}$ is given, one forms the spatial deformation velocity $\mathbf{L}$ via

$$
\begin{aligned}
&\dot{\mathbf{F}} = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}\Big) = \frac{\partial \dot{\mathbf{x}}}{\partial \mathbf{x}_0} = \frac{\partial \dot{\mathbf{x}}}{\partial \mathbf{x}}\,\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0} =: \mathbf{L}\,\mathbf{F} = (\dot{\mathbf{a}}_i \otimes \mathbf{a}^i)(\mathbf{a}_j \otimes \mathbf{h}^j) \\
&\text{with} \quad \mathbf{L} = \frac{\partial \dot{\mathbf{x}}}{\partial \mathbf{x}} = \dot{\mathbf{F}}\,\mathbf{F}^{-1} = (\dot{\mathbf{a}}_i \otimes \mathbf{h}^i)\mathbf{F}^{-1} = \dot{\mathbf{a}}_i \otimes \mathbf{F}^{T-1}\mathbf{h}^i = \dot{\mathbf{a}}_i \otimes \mathbf{a}^i
\end{aligned}
$$

$$
\longrightarrow \quad \boxed{\mathbf{F} = \mathbf{a}_i \otimes \mathbf{h}^i\,, \quad \dot{\mathbf{F}} = \dot{\mathbf{a}}_i \otimes \mathbf{h}^i\,, \quad \mathbf{L} = \dot{\mathbf{a}}_i \otimes \mathbf{a}^i}
$$

As $\mathbf{F}$ is always invertible as a result of $\det \mathbf{F} > 0$, this is not necessarily the case for $\dot{\mathbf{F}}$, for example, under simple shear conditions, where $\det \dot{\mathbf{F}} = 0$. This leads to

$$
\dot{\mathbf{I}} = (\mathbf{F}\mathbf{F}^{-1})^{\boldsymbol{\cdot}} = \underbrace{\dot{\mathbf{F}}\mathbf{F}^{-1}}_{\mathbf{L}} + \underbrace{\mathbf{F}(\mathbf{F}^{-1})^{\boldsymbol{\cdot}}}_{-\mathbf{L}} = \mathbf{0}
$$

Following the above, one observes two possibilities to describe $\mathbf{L}$

$$
\boxed{\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} \quad \text{and} \quad \mathbf{L} = -\mathbf{F}(\mathbf{F}^{-1})^{\boldsymbol{\cdot}}}
$$

Describing these terms by use of the natural basis system yields

$$\begin{aligned} \mathbf{F} &= \mathbf{a}_i \otimes \mathbf{h}^i & \longrightarrow \quad \dot{\mathbf{F}} &= \dot{\mathbf{a}}_i \otimes \mathbf{h}^i \\ \mathbf{F}^{-1} &= \mathbf{h}_i \otimes \mathbf{a}^i & \longrightarrow \quad (\mathbf{F}^{-1})^{\boldsymbol{\cdot}} &= \mathbf{h}_i \otimes \dot{\mathbf{a}}^i \end{aligned}$$

such that

$$\boxed{\mathbf{L} = \begin{cases} \dot{\mathbf{F}}\mathbf{F}^{-1} &= (\dot{\mathbf{a}}_i \otimes \mathbf{a}^i) \\ -\mathbf{F}(\mathbf{F}^{-1})^{\boldsymbol{\cdot}} &= -(\mathbf{a}_i \otimes \dot{\mathbf{a}}^i) \end{cases} \quad \text{and} \quad \mathbf{L}^T = \begin{cases} \mathbf{F}^{T-1}\dot{\mathbf{F}}^T &= (\mathbf{a}^i \otimes \dot{\mathbf{a}}_i) \\ -(\mathbf{F}^{T-1})^{\boldsymbol{\cdot}}\mathbf{F}^T &= -(\dot{\mathbf{a}}^i \otimes \mathbf{a}_i) \end{cases}}$$

With $\mathbf{L}$ and $\mathbf{L}^T$, the rates of $\mathbf{a}_i$ and $\mathbf{a}^i$ can be given as

$$\begin{aligned} \dot{\mathbf{a}}_i &= \mathbf{L}\,\mathbf{a}_i &&= (\dot{\mathbf{a}}_n \otimes \mathbf{a}^n)\,\mathbf{a}_i = \dot{\mathbf{a}}_n \delta_i^n = \dot{\mathbf{a}}_i \\ \dot{\mathbf{a}}^i &= -\mathbf{L}^T\mathbf{a}^i &&= (\dot{\mathbf{a}}^n \otimes \mathbf{a}_n)\,\mathbf{a}^i = \dot{\mathbf{a}}^n \delta_n^i = \dot{\mathbf{a}}^i \end{aligned}$$

Deformation velocity D and spin tensor W: By splitting the spatial velocity gradient $\mathbf{L}$ in a symmetric and a skew-symmetric part, one obtains

$$\mathbf{L} = \mathbf{D} + \mathbf{W}$$

$$\text{where} \begin{cases} \mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T) & \text{with} \quad \mathbf{D} = \mathbf{D}^T \\ \mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T) & \text{with} \quad \mathbf{W} = -\mathbf{W}^T \end{cases}$$

and

$$\boxed{\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T) = \begin{cases} \frac{1}{2}(\dot{\mathbf{a}}_i \otimes \mathbf{a}^i + \mathbf{a}^i \otimes \dot{\mathbf{a}}_i) \\ -\frac{1}{2}(\mathbf{a}_i \otimes \dot{\mathbf{a}}^i + \dot{\mathbf{a}}^i \otimes \mathbf{a}_i) \end{cases}}$$

This leads to the following conclusions

$$\mathbf{0} = \dot{\mathbf{I}} = (a_{ik}\,\mathbf{a}^i \otimes \mathbf{a}^k)^{\boldsymbol{\cdot}} = \underbrace{\dot{a}_{ik}(\mathbf{a}^i \otimes \mathbf{a}^k)}_{2\mathbf{D}} + \underbrace{a_{ik}(\dot{\mathbf{a}}^i \otimes \mathbf{a}^k)}_{\dot{\mathbf{a}}^i \otimes \mathbf{a}_i = -\mathbf{L}^T} + \underbrace{a_{ik}(\mathbf{a}^i \otimes \dot{\mathbf{a}}^k)}_{\mathbf{a}_i \otimes \dot{\mathbf{a}}^i = -\mathbf{L}}$$

$$\mathbf{0} = \dot{\mathbf{I}} = (a^{ik}\,\mathbf{a}_i \otimes \mathbf{a}_k)^{\boldsymbol{\cdot}} = \underbrace{\dot{a}^{ik}(\mathbf{a}_i \otimes \mathbf{a}_k)}_{-2\mathbf{D}} + \underbrace{a^{ik}(\dot{\mathbf{a}}_i \otimes \mathbf{a}_k)}_{\dot{\mathbf{a}}_i \otimes \mathbf{a}^i = \mathbf{L}} + \underbrace{a^{ik}(\mathbf{a}_i \otimes \dot{\mathbf{a}}_k)}_{\mathbf{a}^i \otimes \dot{\mathbf{a}}_i = \mathbf{L}^T}$$

Thus, there are two further possibilities to describe $\mathbf{D}$

$$\boxed{\mathbf{D} = \begin{cases} \frac{1}{2}\,\dot{a}_{ik}(\mathbf{a}^i \otimes \mathbf{a}^k) \\ -\frac{1}{2}\,\dot{a}^{ik}(\mathbf{a}_i \otimes \mathbf{a}_k) \end{cases}}$$

GREEN-LAGRANGE **strain rates:**

$$
\begin{aligned}
\dot{\mathbf{E}} &= \tfrac{1}{2}(\dot{\mathbf{C}} - \mathbf{I}) &&= \tfrac{1}{2}\dot{a}_{ik}(\mathbf{h}^i \otimes \mathbf{h}^k)\\
\dot{\mathbf{A}} &= \tfrac{1}{2}[\,\mathbf{I} - (\mathbf{B}^{-1})^{\cdot}\,] &&= \tfrac{1}{2}\dot{a}_{ij}(\mathbf{a}^i \otimes \mathbf{a}^k) + \tfrac{1}{2}(a_{ik} - h_{ik})[\,(\dot{\mathbf{a}}^i \otimes \mathbf{a}^k) + (\mathbf{a}^i \otimes \dot{\mathbf{a}}^k)\,]\\
& &&= \tfrac{1}{2}\dot{a}_{ij}(\mathbf{a}^i \otimes \mathbf{a}^k) \underbrace{+ \tfrac{1}{2}(a_{ik} - h_{ik})[\,-\mathbf{L}^T\,(\mathbf{a}^i \otimes \mathbf{a}^k) - (\mathbf{a}^i \otimes \mathbf{a}^k)\mathbf{L}^T\,]}_{-\mathbf{L}^T\mathbf{A} - \mathbf{A}\mathbf{L}}
\end{aligned}
$$

Based on the above, it is obvious that the push-forward and pull-back relations between $\mathbf{E}$ and $\mathbf{A}$ do not hold for $\dot{\mathbf{E}}$ and $\dot{\mathbf{A}}$. To overcome this feature, one introduces the so-called contravariant "upper OLDROYD or LIE derivative" yielding

$$
\overset{\triangle}{\mathbf{A}} := \dot{\mathbf{A}} + \mathbf{L}^T\mathbf{A} + \mathbf{A}\mathbf{L} = \mathbf{D} = \tfrac{1}{2}\dot{a}_{ik}(\mathbf{a}^i \otimes \mathbf{a}^k) \quad \longrightarrow \quad \begin{cases} \overset{\triangle}{\mathbf{A}} = \mathbf{F}^{T-1}\,\dot{\mathbf{E}}\,\mathbf{F}^{-1} \\ \dot{\mathbf{E}} = \mathbf{F}^T\overset{\triangle}{\mathbf{A}}\,\mathbf{F} \end{cases}
$$

James Gardner OLDROYD (1921–1982) was a British mathematician and rheologist.
Marius Sophus LIE (1842-1899) wa a Norwegian mathematician.

KARNI-REINER strain rates:

$$
\begin{aligned}
\dot{\mathbf{K}}_R &= \tfrac{1}{2}(\,\mathbf{I} - \dot{\mathbf{C}}^{-1}) &&= -\tfrac{1}{2}\dot{a}^{ik}(\mathbf{h}_i \otimes \mathbf{h}_k)\\
\dot{\mathbf{K}}_C &= \tfrac{1}{2}(\dot{\mathbf{B}} - \mathbf{I}) &&= -\tfrac{1}{2}\dot{a}^{ik}(\mathbf{a}_i \otimes \mathbf{a}_k) + \tfrac{1}{2}(h^{ik} - a^{ik})[\,(\dot{\mathbf{a}}_i \otimes \mathbf{a}_k) + (\mathbf{a}_i \otimes \dot{\mathbf{a}}_k)\,]\\
& &&= -\tfrac{1}{2}\dot{a}^{ik}(\mathbf{a}_i \otimes \mathbf{a}_k) \underbrace{+ \tfrac{1}{2}(h^{ik} - a^{ik})[\,\mathbf{L}\,(\mathbf{a}_i \otimes \mathbf{a}_k) + (\mathbf{a}_i \otimes \mathbf{a}_k)\,\mathbf{L}^T\,]}_{\mathbf{L}\,\mathbf{K}_C + \mathbf{K}_C\mathbf{L}^T}
\end{aligned}
$$

As for the GREEN-LAGRANGE and the ALMANSI strains, there is no push-forward and pull-back relation between $\dot{\mathbf{K}}^R$ and $\dot{\mathbf{K}}^C$ as it is for $\mathbf{K}^R$ and $\mathbf{K}^C$. This leads to the so-called covariant "lower OLDROYD or LIE derivative" for $\mathbf{K}^C$ yielding

$$
\overset{\triangledown}{\mathbf{K}}_C := \dot{\mathbf{K}}_C - \mathbf{L}\mathbf{K}_C - \mathbf{K}_C\mathbf{L}^T = \mathbf{D} = -\tfrac{1}{2}\dot{a}^{ik}(\mathbf{a}_i \otimes \mathbf{a}_k) \quad \longrightarrow \quad \begin{cases} \overset{\triangledown}{\mathbf{K}}_C = \mathbf{F}\,\dot{\mathbf{K}}_R\,\mathbf{F}^T \\ \dot{\mathbf{K}}_R = \mathbf{F}^{-1}\overset{\triangledown}{\mathbf{K}}_C\,\mathbf{F}^{T-1} \end{cases}
$$

B.4 Transport theorems

In this section, the transport theorems for line, area and volume elements will be presented with respect to a natural basis system.

(1) Line elements in the reference configuration at time t_0 and in the current configuration at time t:

$$
\left.\begin{aligned} &\mathrm{d}\mathbf{x}_0\,(t_0) = \mathrm{d}\theta^1\mathbf{h}_1\,(t_0)\,, \quad \mathrm{d}\mathbf{x}\,(t) = \mathrm{d}\theta^1\mathbf{a}_1(t)\\ &\mathbf{a}_i = \mathbf{F}\,\mathbf{h}_i \quad \longrightarrow \quad \mathrm{d}\mathbf{x} = \mathrm{d}\theta^1\mathbf{F}\,\mathbf{h}_1 = \mathrm{d}\theta^1\mathbf{a}_1 \end{aligned}\right\} \longrightarrow \quad \boxed{\mathrm{d}\mathbf{x} = \mathbf{F}\,\mathrm{d}\mathbf{x}_0}
$$

(2) Volume elements in the reference configuration at time t_0 and in the current configuration at time t:

$$\begin{aligned} \mathrm{d}v_0\,(t_0) &= \mathrm{d}\theta^1\mathrm{d}\theta^2\mathrm{d}\theta^3\,\underbrace{(\mathbf{h}_1\times\mathbf{h}_2)}_{\sqrt{h}\,\mathbf{h}^3}\cdot\mathbf{h}_3 = \mathrm{d}\theta^1\mathrm{d}\theta^2\mathrm{d}\theta^3\sqrt{h} \\ \mathrm{d}v\,(t) &= \mathrm{d}\theta^1\mathrm{d}\theta^2\mathrm{d}\theta^3\,\underbrace{(\mathbf{a}_1\times\mathbf{a}_2)}_{\sqrt{a}\,\mathbf{a}^3}\cdot\mathbf{a}_3 = \mathrm{d}\theta^1\mathrm{d}\theta^2\mathrm{d}\theta^3\sqrt{a} \end{aligned}$$

$$\longrightarrow \quad \boxed{\mathrm{d}v = \frac{\sqrt{a}}{\sqrt{h}}\,\mathrm{d}v_0}$$

with $a = \det|a_{ik}|$ and $h = \det|h_{ik}|$. To find the meaning of $\sqrt{a}/\sqrt{h}$, one proceeds with the determinant of the right Cauchy-Green deformation tensor $\mathbf{C}$:

$$\det\mathbf{C} = \det(\mathbf{F}^T\mathbf{F}) = (\det\mathbf{F})^2$$

From the above computation of the determinant of an arbitrary tensor, compare p. 58, one concludes to

$$\det\mathbf{C} = \det\left[\,a_{ik}(\mathbf{h}^i\otimes\mathbf{h}^k)\,\right] = (\det|a_{ik}|)\,(\det|h^{ik}|) = (\det|a_{ik}|)\,(\det|h_{ik}|)^{-1} = \frac{a}{h}$$

$$\longrightarrow \quad \boxed{\det\mathbf{F} = \frac{\sqrt{a}}{\sqrt{h}} \quad \longrightarrow \quad \mathrm{d}v = \det\mathbf{F}\,\mathrm{d}v_0}$$

(3) Area elements in the reference configuration at time t_0 and in the current configuration at time t with, for example, directions $\mathbf{h}^3$ at t_0 and $\mathbf{a}^3$ at t:

$$\begin{aligned} \mathrm{d}\mathbf{a}_0\,(t_0) &= \mathrm{d}\theta^1\mathrm{d}\theta^2(\mathbf{h}_1\times\mathbf{h}_2) = \mathrm{d}\theta^1\mathrm{d}\theta^2\sqrt{h}\,\mathbf{h}^3 \\ \mathrm{d}\mathbf{a}\,(t) &= \mathrm{d}\theta^1\mathrm{d}\theta^2(\mathbf{a}_1\times\mathbf{a}_2) = \mathrm{d}\theta^1\mathrm{d}\theta^2\sqrt{a}\,\mathbf{a}^3 \end{aligned}$$

where, for example, $\mathbf{a}^3$ can be given as

$$\mathbf{a}^3 = \mathbf{F}^{T-1}\,\mathbf{h}^3 = \frac{\operatorname{cof}\mathbf{F}}{\det\mathbf{F}}\,\mathbf{h}^3 = \frac{\sqrt{h}}{\sqrt{a}}\,(\operatorname{cof}\mathbf{F})\,\mathbf{h}^3$$

$$\longrightarrow \quad \boxed{\mathrm{d}\mathbf{a} = \mathrm{d}\theta^1\mathrm{d}\theta^2\sqrt{h}\,(\operatorname{cof}\mathbf{F})\,\mathbf{h}^3 \quad \longrightarrow \quad \mathrm{d}\mathbf{a} = (\operatorname{cof}\mathbf{F})\,\mathrm{d}\mathbf{a}_0}$$

C Stress and stress power

C.1 Cauchy, Kirchhoff and Piola-Kirchhoff stresses

Stresses $\mathbf{t}$ are exerted to a material body $\mathcal{B}$ by external forces acting at the surface $\mathcal{S}$ yielding

$$\int_{\mathcal{S}} \mathbf{t}\,\mathrm{d}a = \int_{\mathcal{S}} \mathbf{T}\,\mathbf{n}\,\mathrm{d}a = \int_{\mathcal{S}} \mathbf{T}\,\mathrm{d}\mathbf{a} = \int_{\mathcal{B}} \operatorname{div}\mathbf{T}\,\mathrm{d}v$$

Therein, the scalar surface element $\mathrm{d}a$ of the current configuration is transformed into a vector-valued surface element $\mathrm{d}\mathbf{a} = \mathbf{n}\,\mathrm{d}a$ with $\mathbf{n}$ as the outward-oriented unit surface normal. Furthermore, use has been made of the CAUCHY theorem $\mathbf{t} = \mathbf{T}\,\mathbf{n}$ with $\mathbf{T}$ as the CAUCHY stress, also called true stress. Finally, an integral theorem transfers the tensor-valued function $\mathbf{T}$ at the oriented surface with surface element $\mathrm{d}\mathbf{a}$ towards a vector-valued function $\operatorname{div}\mathbf{T}$ in the body $\mathcal{B}$ with volume element $\mathrm{d}v$, compare Section 3.4.(d).

Surface element, for example $\mathrm{d}\mathbf{a}^3$, surface normal and volume element:

$$\begin{aligned}
\mathrm{d}\mathbf{a}^3 &= \mathrm{d}\mathbf{x}_1 \times \mathrm{d}\mathbf{x}_2 = \mathrm{d}\theta^1\mathbf{a}_1 \times \mathrm{d}\theta^2\mathbf{a}_2 = \mathrm{d}\theta^1\mathrm{d}\theta^2\sqrt{a}\,\mathbf{a}^3 \quad\longrightarrow\quad \mathbf{n}^3 = \mathrm{d}\mathbf{a}^3/|\mathrm{d}\mathbf{a}^3| = \mathbf{a}^3/\sqrt{a^{33}} \\
\mathrm{d}v &= (\mathrm{d}\mathbf{x}_1 \times \mathrm{d}\mathbf{x}_2)\cdot\mathrm{d}\mathbf{x}_3 = \mathrm{d}\theta^1\mathrm{d}\theta^2\sqrt{a}\,\mathbf{a}^3\cdot\mathrm{d}\theta^3\mathbf{a}_3 = \mathrm{d}\theta^1\mathrm{d}\theta^2\mathrm{d}\theta^3\sqrt{a}
\end{aligned}$$

with $\sqrt{a^{33}} = \operatorname{cof} a_{33}/(\det A)$ and $\det A = \det|a_{ik}| = a$, compare p. 56

Stress vector and stress tensor:

For the present example, the normal of the surface under consideration is again oriented towards the $\mathbf{a}^3$ direction. Thus, with

$$\begin{aligned}
&\mathbf{T} = t^{ik}(\mathbf{a}_i \otimes \mathbf{a}_k) \\
&\text{where}\quad \mathbf{t} = \mathbf{T}\,\mathbf{n}^3 = t^{ik}(\mathbf{a}_i \otimes \mathbf{a}_k)\frac{\mathbf{a}^3}{\sqrt{a^{33}}} = \frac{t^{i3}}{\sqrt{a^{33}}}\,\mathbf{a}_i
\end{aligned}$$

Cauchy, Kirchhoff and Piola-Kirchhoff stresses

$$\mathbf{T}\,\mathrm{d}\mathbf{a} = \mathbf{T}\,(\operatorname{cof}\mathbf{F})\,\mathrm{d}\mathbf{a}_0 = (\det\mathbf{F})\,\mathbf{T}\,\mathbf{F}^{T-1}\mathrm{d}\mathbf{a}_0$$

$$\text{where}\quad \begin{cases} \boldsymbol{\tau} = (\det\mathbf{F})\,\mathbf{T} & : \ \text{Kirchhoff stress} \\ \mathbf{P} = (\det\mathbf{F})\,\mathbf{T}\,\mathbf{F}^{T-1} & : \ \text{1st Piola-Kirchhoff stress} \end{cases}$$

$$\text{in addition:}\quad \mathbf{S} = (\det\mathbf{F})\,\mathbf{F}^{-1}\mathbf{T}\,\mathbf{F}^{T-1} : \ \text{2nd Piola-Kirchhoff stress}$$

Gabriel PIOLA (1794-1850) was an Italian mathematician, who did not accept a professorial offer of a university. Instead, he preferred to live as private tutor.

Gustav Robert KIRCHHOFF (1824-1887) was a German physicist working on electricity problems. He held professorships at Breslau (nowadays Wrocław), Heidelberg and Berlin.

Concerning the above stresses, the KIRCHHOFF stress $\boldsymbol{\tau}$ is the volumetrically weighted CAUCHY stress what can be seen by the inclusion of $\det \mathbf{F} = \mathrm{d}v/\mathrm{d}v_0 = \sqrt{a}/\sqrt{h}$. Thus,

$$\boldsymbol{\tau} = \frac{\sqrt{a}}{\sqrt{h}}\, t^{ik} (\mathbf{a}_i \otimes \mathbf{a}_k) =: \tau^{ik}\, (\mathbf{a}_i \otimes \mathbf{a}_k)$$

While CAUCHY and KIRCHHOFF stresses are symmetric acting at the current configuration, the 1st PIOLA-KIRCHHOFF stress $\mathbf{P} = \boldsymbol{\tau}\,\mathbf{F}^{T-1}$, also called nominal stress, is non-symmetric as a result of its two-field character with the first basis in the current and the second in the reference configuration:

$$\mathbf{P} = \frac{\sqrt{a}}{\sqrt{h}}\, t^{ik}\, (\mathbf{a}_i \otimes \mathbf{a}_k)\, (\mathbf{a}^n \otimes \mathbf{h}_n) = \frac{\sqrt{a}}{\sqrt{h}}\, t^{ik}\, (\mathbf{a}_i \otimes \mathbf{h}_k) = \tau^{ik}\, (\mathbf{a}_i \otimes \mathbf{h}_k)$$

$$\text{and} \quad \begin{cases} \mathbf{T}\,\mathrm{d}\mathbf{a}^3 = t^{ik}\, (\mathbf{a}_i \otimes \mathbf{a}_k)\, \mathrm{d}\theta^1 \mathrm{d}\theta^2 \sqrt{a}\, \mathbf{a}^3 = t^{i3}\, \mathrm{d}\theta^1 \mathrm{d}\theta^2 \sqrt{a}\, \mathbf{a}_i \\ \mathbf{P}\,\mathrm{d}\mathbf{a}_0^3 = \dfrac{\sqrt{a}}{\sqrt{h}}\, t^{ik}\, (\mathbf{a}_i \otimes \mathbf{h}_k)\, \mathrm{d}\theta^1 \mathrm{d}\theta^2 \sqrt{h}\, \mathbf{h}^3 = t^{i3}\, \mathrm{d}\theta^1 \mathrm{d}\theta^2 \sqrt{a}\, \mathbf{a}_i \end{cases}$$

From the above equations, it is seen that the stress vector $\mathbf{t} = \mathbf{T}\,\mathrm{d}\mathbf{a}^3$ exerted on the current surface element is identical compared with the same values and the same directions of the stress $\mathbf{P}\,\mathrm{d}\mathbf{a}_0^3$ exerted on the reference surface element both proceeding from the current directions $\mathbf{a}_i$.

However, concerning the load vector $\mathbf{p} = \mathbf{P}\,\mathbf{n}_0^3$, things are different resulting in

$$\mathbf{p} = \mathbf{P}\,\mathbf{n}_0^3 = \frac{\sqrt{a}}{\sqrt{h}}\, \frac{t^{i3}}{\sqrt{h^{33}}}\, \mathbf{a}_i \quad \longrightarrow \quad \mathbf{p} = \left(\frac{\sqrt{a}}{\sqrt{h}}\, \frac{\sqrt{a^{33}}}{\sqrt{h^{33}}} \right) \frac{t^{i3}}{\sqrt{a^{33}}}\, \mathbf{a}_i = \left(\frac{\sqrt{a}}{\sqrt{h}}\, \frac{\sqrt{a^{33}}}{\sqrt{h^{33}}} \right) \mathbf{t}$$

where $\mathrm{d}\mathbf{a}_0^3 = \mathrm{d}\theta^1 \mathrm{d}\theta^2 \sqrt{h}\, \mathbf{h}^3$ together with $\mathbf{n}_0^3 = \mathbf{h}^3/\sqrt{h^{33}}$ has been used.

Finally, writing the 2nd PIOLA-KIRCHHOFF stress in a natural basis system yields

$$\mathbf{S} = \mathbf{F}^{-1} \boldsymbol{\tau}\, \mathbf{F}^{T-1} = (\mathbf{h}_j \otimes \mathbf{a}^j) \tau^{ik}\, (\mathbf{a}_i \otimes \mathbf{a}_k)\, (\mathbf{a}^l \otimes \mathbf{h}_l) = \tau^{ik}\, (\mathbf{h}_i \otimes \mathbf{h}_k)$$

Thus, the 2nd PIOLA-KIRCHHOFF stress $\mathbf{S}$ can be understood as a formal pull back of the KIRCHHOFF stress towards the reference configuration. However, $\mathbf{S}$ is not directly connected with the load vector $\mathbf{p}$ but with an artficial load vector

$$\bar{\mathbf{p}} = \mathbf{F}^{-1} \mathbf{p} = \left(\frac{\sqrt{a}}{\sqrt{h}}\, \frac{\sqrt{a^{33}}}{\sqrt{h^{33}}} \right) \frac{t^{i3}}{\sqrt{a^{33}}}\, \mathbf{h}_i$$

where the basis vectors $\mathbf{a}_i$ have been exchanged by $\mathbf{h}_i$.

C.2 Stress power

Mechanical energy is exerted onto a body by

$$\begin{aligned}\int_{\mathcal{S}} \mathbf{t} \cdot \dot{\mathbf{x}} \, \mathrm{d}a &= \int_{\mathcal{S}} \mathbf{T}\,\mathbf{n} \cdot \dot{\mathbf{x}} \, \mathrm{d}a = \int_{\mathcal{S}} \mathbf{T}^T \dot{\mathbf{x}} \cdot \mathrm{d}\mathbf{a} \\ &= \int_{\mathcal{B}} (\operatorname{div} \mathbf{T} \cdot \dot{\mathbf{x}} + \mathbf{T} \cdot \operatorname{grad} \dot{\mathbf{x}}) \, \mathrm{d}v = \int_{\mathcal{B}} \mathbf{T} \cdot \mathbf{L} \, \mathrm{d}v\end{aligned}$$

Therein, static equilibrium without body forces through $\operatorname{div} \mathbf{T} = \mathbf{0}$ has been assumed. The stress power $\mathbf{T} \cdot \mathbf{L}$ can now be displayed in various forms, where the symmetry of $\mathbf{T}$ can be used:

$$\mathbf{T} \cdot \mathbf{L} = \mathbf{T} \cdot \mathbf{D} = \det \mathbf{F} \, (\boldsymbol{\tau} \cdot \mathbf{D}) = \det \mathbf{F} \, (\boldsymbol{\tau} \cdot \overset{\triangle}{\mathbf{A}})$$

Here, the equivalence of $\mathbf{D}$ and $\overset{\triangle}{\mathbf{A}}$ has been taken into consideration. Proceeding from the pull-back and push-forward operations of stresses and strain rates, one obtains

$$\left.\begin{aligned}\boldsymbol{\tau} \cdot \mathbf{D} &= \mathbf{F}\,\mathbf{S}\,\mathbf{F}^T \cdot \mathbf{F}^{T-1} \dot{\mathbf{E}}\,\mathbf{F}^{-1} = \mathbf{S} \cdot \dot{\mathbf{E}} \\ \boldsymbol{\tau} \cdot \mathbf{L} &= \mathbf{P}\,\mathbf{F}^T \cdot \dot{\mathbf{F}}\mathbf{F}^{-1} = \mathbf{P} \cdot \dot{\mathbf{F}}\end{aligned}\right\} \longrightarrow \quad \mathbf{S} \cdot \dot{\mathbf{E}} = \mathbf{P} \cdot \dot{\mathbf{F}}$$

With respect to a natural basis system, the above relations yield

$$\begin{aligned}\boldsymbol{\tau} \cdot \mathbf{D} &= \tau^{ik} \, (\mathbf{a}_i \otimes \mathbf{a}_k) \cdot \tfrac{1}{2} \, \dot{\mathrm{a}}_{no} \, (\mathbf{a}^n \otimes \mathbf{a}^o) = \tfrac{1}{2} \, \tau^{ik} \, \dot{\mathrm{a}}_{ik} \\ \mathbf{S} \cdot \dot{\mathbf{E}} &= \tau^{ik} \, (\mathbf{h}_i \otimes \mathbf{h}_k) \cdot \tfrac{1}{2} \, \dot{\mathrm{a}}_{no} \, (\mathbf{h}^n \otimes \mathbf{h}^o) = \tfrac{1}{2} \, \tau^{ik} \, \dot{\mathrm{a}}_{ik}\end{aligned}$$

Using $\mathbf{D} = \frac{1}{2}(\dot{\mathbf{a}}_i \otimes \mathbf{a}^i + \mathbf{a}^i \otimes \dot{\mathbf{a}}_i)$, $\boldsymbol{\tau} \cdot \mathbf{D}$ reads

$$\begin{aligned}\boldsymbol{\tau} \cdot \mathbf{D} &= \tau^{ik} \, (\mathbf{a}_i \otimes \mathbf{a}_k) \cdot \tfrac{1}{2}(\dot{\mathbf{a}}_s \otimes \mathbf{a}^s + \mathbf{a}^s \otimes \dot{\mathbf{a}}_s) \\ &= \tfrac{1}{2} \, \tau^{ik} [\, (\mathbf{a}_i \cdot \dot{\mathbf{a}}_s) \delta^s_k + \delta^s_i (\mathbf{a}_k \cdot \dot{\mathbf{a}}_s) \,] = \tfrac{1}{2} \, \tau^{ik} [\, (\mathbf{a}_i \cdot \dot{\mathbf{a}}_k) + (\mathbf{a}_k \cdot \dot{\mathbf{a}}_i) \,]\end{aligned}$$

To prove that this result is equivalent with $\boldsymbol{\tau} \cdot \mathbf{L}$, use is made of the symmetry of $\boldsymbol{\tau}$. Thus,

$$\begin{aligned}\boldsymbol{\tau} \cdot \mathbf{L} &= \tfrac{1}{2}(\boldsymbol{\tau} + \boldsymbol{\tau}^T) \cdot \mathbf{L} = \tfrac{1}{2}(\tau^{ik} + \tau^{ki}) \, (\mathbf{a}_i \otimes \mathbf{a}_k) \cdot (\dot{\mathbf{a}}_s \otimes \mathbf{a}^s) \\ &= \tfrac{1}{2}(\tau^{ik} + \tau^{ki}) \, (\mathbf{a}_i \cdot \dot{\mathbf{a}}_s) \delta^s_k = \tfrac{1}{2}(\tau^{ik} + \tau^{ki}) \, (\mathbf{a}_i \cdot \dot{\mathbf{a}}_k) \\ &= \tfrac{1}{2}(\tau^{ik} + \tau^{ki}) \, (\mathbf{a}_i \cdot \dot{\mathbf{a}}_k) = \tfrac{1}{2} \, \tau^{ik} [\, (\mathbf{a}_i \cdot \dot{\mathbf{a}}_k) + (\mathbf{a}_k \cdot \dot{\mathbf{a}}_i) \,] \qquad \text{q. e. d.}\end{aligned}$$

Without using the symmetry of the Kirchhoff stress, $\boldsymbol{\tau} \cdot \mathbf{L}$ yields the same result as $\mathbf{P} \cdot \dot{\mathbf{F}}$, what can easily be seen from

$$\begin{aligned}\boldsymbol{\tau} \cdot \mathbf{L} &= \tau^{ik} \, (\mathbf{a}_i \otimes \mathbf{a}_k) \cdot (\dot{\mathbf{a}}_s \otimes \mathbf{a}^s) = \tau^{ik} \, (\mathbf{a}_i \cdot \dot{\mathbf{a}}_k) \\ \mathbf{P} \cdot \dot{\mathbf{F}} &= \tau^{ik} \, (\mathbf{a}_i \otimes \mathbf{h}_k) \cdot (\dot{\mathbf{a}}_s \otimes \mathbf{h}^s) = \tau^{ik} \, (\mathbf{a}_i \cdot \dot{\mathbf{a}}_k)\end{aligned}$$